Some Properties of Fundamental Linear Canonical Zak Transform

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Abstract

In this paper, we introduce some fundamentally properties of Zak linear canonical transform (LCZT) such as linearity and translation properties. LCZT is developing of Zak transform (ZT) and linear canonical transform (LCT).

Keywords: linear canonical Zak transform; Zak transform; linear canonical transform; linearity property; translation property.

Abstrak

Dalam jurnal ini akan diungkapkan beberapa sifat fundamental dari transformasi Zak linear kanonik (LCZT), yaitu sifat linear dan sifat translasi. LCZT merupakan hasil pengembangan dari dua buah transformasi yaitu transformasi Zak (ZT) dan transformasi linear kanonik (LCT).

Kata kunci: Transformasi Zak linear kanonik; transformasi Zak; transformasi linear kanonik; sifat linear; sifattranslasi.

1. Introduction

Definition of Zak transformation was be introduced in 1967 by J. Zak [4] that stated as follows.

Definisi1.1[3,4]

Zak transformation $Zf(t,\omega)$ of a function $f(t) \in L^2(\mathbb{R})$ is defined as

$$\tilde{f}(t,\omega) = (\mathcal{Z}_a f)(t,\omega) = \sqrt{a} \sum_{k \in \mathbb{Z}} f(at - ak) e^{-2\pi i k \omega} \tag{1.1}$$

where $a > 0$ and t, ω be real number parameters.

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Without loss of generality, by taking $a = 1$ then from (1.1) we get

$$(\mathcal{Z}f)(t, \omega) = \sum_{k \in \mathbb{Z}} f(t - k) e^{-2\pi i k \omega}. \tag{1.2}$$

Some properties that hold in Zak transformation are given as follows

Theorem 1.1[3,4]

1. *Linearity.*

   Zak transformation is linear, namely for real number a and b this equation holds

   $$[\mathcal{Z}(af + bg)](t, \omega) = a\tilde{f}(t, \omega) + b\tilde{g}(t, \omega). \tag{1.3}$$

2. *Translation.*

   $$\left.\begin{array}{l} [\mathcal{Z}(T_a f)](t, \omega) = \tilde{f}(t - a, \omega), \\ [\mathcal{Z}(T_{-m} f)](t, \omega) = e^{2\pi i m \omega} \tilde{f}(t, \omega) \end{array}\right]. \tag{1.4}$$

3. *Modulation.*

   $$\left.\begin{array}{l} [\mathcal{Z}(M_b f)](t, \omega) = e^{ibt} \tilde{f}\left(t, \omega - \dfrac{b}{2\pi}\right), \\ [\mathcal{Z}(M_{2\pi b} f)](t, \omega) = e^{2\pi ibt} \tilde{f}(t, \omega - b). \end{array}\right]. \tag{1.5}$$

4. *Translation and Modulation.*

   $$[\mathcal{Z}(M_{2\pi b} T_n f)](t, \omega) = e^{2\pi i(mt - n\omega)} \tilde{f}(t, \omega). \tag{1.6}$$

5. *conjugation.*

   $$\left[\mathcal{Z}(\bar{f})\right](t, \omega) = \overline{\tilde{f}(t, -\omega)}. \tag{1.7}$$

6. *Symmetric.*

   (a). If f is even function, then

   $$(\mathcal{Z}f)(t, \omega) = \tilde{f}(-t, -\omega). \tag{1.8}$$

   (b). If f is odd function, then

   $$(\mathcal{Z}f)(t, \omega) = -\tilde{f}(-t, -\omega). \tag{1.9}$$

7. *Inversion.*

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$$f(t) = \int_0^1 \tilde{f}(t,\omega)d\omega,$$

$$\hat{f}(\omega) = \int_0^1 e^{-2\pi i\omega t}\tilde{f}(t,\omega)dt,$$

$$f(\omega) = \int_0^1 e^{-2\pi i\omega t}\tilde{f}(t,\omega)dt. \qquad (1.10)$$

8.   *Dilatation.*

$$\left(ZD_{\frac{1}{a}}f\right)(t,\omega) = \tilde{f}\left(at,\frac{\omega}{a}\right). \qquad (1.11)$$

9.   *Convolution.*

$$\tilde{f}(t,\omega)\overline{\tilde{f}(t,\omega)} = \sum_{m,n=-\infty}^{\infty} \langle f, M_{2\pi m}T_{-n}g\rangle e^{2\pi i(mt+n\omega)}. \qquad (1.12)$$

Teorema 1.2 [3]
Zak transformation is a symmetric mapping of $L^2(\mathbb{R})$ ke $L^2([0,1]^2)$.

In 1970 Linear Canonic Transformation (LCT) was firstly proposed by Moshinsky and Collins [5]. Next, the definition of LCT as follows

Definisi1.2 [2]
Let $A = (a,b,c,d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix of parameter that satisfies $\det(A) = ad - bc = 1$, LCT of a signal $f(t) \in L^1(\mathbb{R})$ be given as

$$\mathcal{L}_A\{f\}(\omega) = \begin{cases} \int_{-\infty}^{\infty} f(t)K_A(\omega,t)dt, & b \neq 0 \\ \sqrt{d}e^{i\frac{cd}{2}\omega^2}f(d\omega), & b = 0 \end{cases} \qquad (1.13)$$

Where $K_A(\omega,t)$ is kernel of LCT that is given as follows

$$K_A(\omega,t) = \frac{1}{\sqrt{2\pi bi}} e^{i\frac{1}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}t\omega + \frac{d}{b}\omega^2\right)}. \qquad (1.14)$$

It is easy to show that kernel of LCT satisfies the following properties

$$K_{A^{-1}}(\omega,t) = \frac{1}{\sqrt{-2\pi bi}} e^{-i\frac{1}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}t\omega + \frac{d}{b}\omega^2\right)}. \qquad (1.15)$$

From definition 1.2, it obvious that if $b = 0$ then LCT is a product of $\sqrt{d}e^{i\frac{cd}{2}\omega^2}$ and $d\omega$
Therefore, we only use for $b \neq 0$.
In the particularly case, if $A = (0,1,-1,0)$ then definition of LCT can be derived to be Fourier transformation, namely

$$\mathcal{L}_A\{f\}(\omega) = \mathcal{F}_A\{f\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-it\omega}dt. \qquad (1.16)$$

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Teorema1.3 [3]

Let $f, g \in L^1(\mathbb{R})$ then the following properties hold

1. *Translation.*

   For all real number k then

$$\mathcal{L}_A\{\tau_k f\}(\omega) = \mathcal{L}_A\{f(t-k)\}(\omega) = e^{\frac{-iack^2}{2}+ick\omega}\mathcal{F}_A(\omega - ak).\tag{1.17}$$

2. *Modulation.*

   For all real number ω_0 then

$$\mathcal{L}_A\{M_{\omega_0}f\}(\omega) = \mathcal{L}_A\{e^{it\omega_0}f(t)\}(\omega) = e^{-\frac{ibd\omega_0^2}{2}+id\omega_0\omega}\mathcal{F}_A(\omega - b\omega_0).\tag{1.18}$$

3. *Translation and modulation.*

   For all real number k and ω_0 then

$$\mathcal{L}_A\{M_{\omega_0}\tau_k f\}(\omega) = \mathcal{L}_A\{e^{ix\omega_0}f(x-k)\}(\omega).\tag{1.19}$$

2.   Linear canonical Zak transformation

Liner Canonical Zak Transformation (LCZT) firstly was introduced by Bahri and Ashino [1,2].

Definition 2.1[1]

LCZT of a signal $f \in L^2(\mathbb{R})$ at point $(t, \omega) \in \mathbb{R} \times \mathbb{R}$ is

$$(\mathcal{Z}_A f)(x, \omega) = \frac{1}{\sqrt{2\pi bi}}\sum_{k\in\mathbb{Z}}f(t-k)e^{i\pi\left(\frac{a}{b}k^2-\frac{2}{b}k\omega+\frac{d}{b}\omega^2\right)}.\tag{2.1}$$

Where $A = (a,b,c,d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times2}$ is a parameter matrix and

$$det(A) = ad - bc = 1.\tag{2.2}$$

If we choose $A_Z = (0,1,-1,0)$ then LCZT can be derived to become definition of Zak transformation with multiplier $\frac{1}{\sqrt{2\pi i}}$ as follows

$$\left(\mathcal{Z}_{A_Z}f\right)(x,\omega) = \frac{1}{\sqrt{2\pi i}}\sum_{k\in\mathbb{Z}}f(t-k)e^{2k\omega\pi i}\tag{2.3}$$

Remarks: If from kernel LCT $\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}\left(\frac{a}{b}t^2-\frac{2}{b}t\omega+\frac{d}{b}\omega^2\right)}$ multiplier $\frac{1}{2}$ replaced by π and variable t replaced by k, then we obtain kernel of LCZT, namely $\frac{1}{\sqrt{2\pi bi}}e^{i\pi\left(\frac{a}{b}k^2-\frac{2}{b}k\omega+\frac{d}{b}\omega^2\right)}$.

Theorem 2.2 (linear property).

If $f, g \in L^2(\mathbb{R})$ and let $A = (a,b,c,d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times2}$, then for all $\alpha, \beta \in \mathbb{C}$

$$\mathcal{Z}_A(\alpha f + \beta g)(t, \omega) = \alpha\mathcal{Z}_A f(t,\omega) + \beta\mathcal{Z}_A g(t,\omega)\tag{2.4}$$

Proof.

From definition 2.1 we have

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$$\mathcal{Z}_A(\alpha f + \beta g)(t, \omega)$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{k \in \mathbb{Z}} (\alpha f + \beta g)(t - k) e^{\frac{i\pi}{b}(ak^2 - 2k\omega + d\omega^2)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{k \in \mathbb{Z}} (\alpha f(t - k) + \beta g(t - k)) e^{\frac{i\pi}{b}(ak^2 - 2k\omega + d\omega^2)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{k \in \mathbb{Z}} \left[\alpha f(t - k) e^{\frac{i\pi}{b}(ak^2 - 2k\omega + d\omega^2)} + \beta g(t - k) e^{\frac{i\pi}{b}(ak^2 - 2k\omega + d\omega^2)} \right]$$

$$= \alpha \frac{1}{\sqrt{2\pi bi}} \sum_{k \in \mathbb{Z}} f(t - k) e^{\frac{i\pi}{b}(ak^2 - 2k\omega + d\omega^2)} + \beta \frac{1}{\sqrt{2\pi bi}} \sum_{k \in \mathbb{Z}} g(t - k) e^{\frac{i\pi}{b}(ak^2 - 2k\omega + d\omega^2)}$$

$$= \alpha \mathcal{Z}_A f(t, \omega) + \beta \mathcal{Z}_A g(t, \omega). \blacksquare$$

Theorem 2.3 (translation property).

If $\in L^2(\mathbb{R})$ and $A = (a, b, c, d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2}$, then for all $m \in \mathbb{Z}$

$$\mathcal{Z}_A(T_m f)(t, \omega) = e^{-\pi i c(2m\omega + cm^2)} \mathcal{Z}_A f(t, \omega + am) \qquad (2.5)$$

where $T_m f(t) = f(t - m)$.

Proof.
Claim that

$$\mathcal{Z}_A(T_m f)(t, \omega) = e^{-\pi i c(2m\omega + cm^2)} \mathcal{Z}_A f(t, \omega + am).$$

From definition2., we obtain

$$\mathcal{Z}_A(T_m f)(x, \omega)$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{k \in \mathbb{Z}} f((t - k) - m) e^{\frac{i\pi}{b}(ak^2 - 2k\omega + d\omega^2)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{k \in \mathbb{Z}} f(t - (k + m)) e^{\frac{i\pi}{b}\left(a((k+m)-m)^2 - 2((k+m)-m)\omega + d\omega^2\right)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{k \in \mathbb{Z}} f(t - k) e^{\frac{i\pi}{b}(a(k-m)^2 - 2(k-m)\omega + d\omega^2)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{l \in \mathbb{Z}} f(t - l) e^{\frac{i\pi}{b}\left(a(l - 2ml + m^2) - 2(l-m)\omega + d\omega^2\right)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{l \in \mathbb{Z}} f(t - l) e^{\frac{i\pi}{b}(al^2 - 2aml + am^2 - 2l\omega + 2m\omega + d\omega^2)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{l \in \mathbb{Z}} f(t - l) e^{\frac{i\pi}{b}(al^2 - 2l(\omega + am) + d(\omega + am)^2 - 2dam\omega - da^2m^2 + am^2 + 2m\omega)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{l \in \mathbb{Z}} f(t - l) e^{\frac{i}{2b}(al^2 - 2l(\omega + am) + d(\omega + am)^2)} e^{\frac{i\pi}{b}(-2dam\omega - da^2m^2 + am^2 + 2m\omega)}$$

$$= \frac{1}{\sqrt{2\pi bi}} \sum_{l \in \mathbb{Z}} f(t - l) e^{\frac{i\pi}{b}(al^2 - 2l(\omega + am) + d(\omega + am)^2)} e^{\frac{i\pi}{b}(-2dam\omega - da^2m^2 + am^2 + 2m\omega)}$$

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$$= e^{\frac{i\pi}{b}(-2dam\omega-da^2m^2+am^2+2m\omega)}\left(\frac{1}{\sqrt{2\pi bi}}\sum_{l\in\mathbb{Z}}f(t-l)e^{\frac{i\pi}{b}(al^2-2l(\omega+am)+d(\omega+am)^2)}\right)$$

$$= e^{\frac{i\pi}{b}(-2dam\omega-da^2m^2+am^2+2m\omega)}\mathcal{Z}_Af(t,\omega+am)$$

$$= e^{\frac{i\pi}{b}(-2dam\omega+2m\omega)}e^{\frac{i\pi}{b}(-da^2m^2+am^2)}\mathcal{Z}_Af(t,\omega+am)$$

$$= e^{-2i\pi\left(\frac{da-1}{b}\right)m\omega}e^{-i\pi\left(\frac{da-1}{b}\right)m^2}\mathcal{Z}_Af(t,\omega+am)$$

$$= e^{-2\pi i\left(\frac{da-1}{b}\right)m\omega}e^{-i\pi\left(\frac{da-1}{b}\right)m^2}\mathcal{Z}_Af(t,\omega+am).$$

Since $ad-bc=1$ then $\frac{da-1}{b}=c$, thus we have

$$\mathcal{Z}_A(T_mf)(t,\omega)=e^{-2\pi icm\omega}e^{-i\pi cm^2}\mathcal{Z}_Af(t,\omega+am)=e^{-\pi ic(2m\omega+cm^2)}\mathcal{Z}_Af(t,\omega+am).\ \blacksquare$$

3.   Conclusion

In LCZT if $f,g\in L^2(\mathbb{R})$ and $A=(a,b,c,d)=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathbb{R}^{2\times2}$ then for all $\alpha,\beta\in\mathbb{C}$ we get

$$\mathcal{Z}_A(\alpha f+\beta g)(t,\omega)=\alpha\mathcal{Z}_Af(t,\omega)+\beta\mathcal{Z}_Ag(t,\omega),$$

and for translation property in LCZT, if $f\in L^2(\mathbb{R})$ and $A=(a,b,c,d)=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathbb{R}^{2\times2}$ then for all $m\in\mathbb{Z}$ hold

$$\mathcal{Z}_A(T_mf)(t,\omega)=e^{-\pi ic(2m\omega+cm^2)}\mathcal{Z}_Af(t,\omega+am).$$

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