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Controlled g-frames and their dual in Hilbert C^*-modules

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Abstract.

In this paper we give some new results for controlled g-frames and controlled dual g-frames in Hilbert C^*-modules. First, we talk about controlled g-frame characterisation and find certain conditions that are equal to them. Then, we explain the purpose controlled dual g-frames and controlled dual g-frames operator and discuss some of their characteristics.

Keywords: g-frame, controlled g-frame, C^*-algebras, Hilbert C^*-modules.

1. Introduction

The notion of frame is a recent active mathematical research topic, signal processing, computer science, etc. Frames for Hilbert spaces were first introduced in **1952** by Duffin and Schaefer [5] for study of nonharmonic Fourier series. Daubechies, Grossmann, and Meyer [4] revived and developed them in **1986**, and popularized from then on.

In recent years, many mathematicians generalized the frame theory from Hilbert spaces to Hilbert C^*-modules. For more details of frames in Hilbert C^*-modules we refer to [8, 10, 13, 7]. Currently, the study of g-frames has yielded many results. Controlled frames have been introduced by Balazs et al.[1] to improve the numerical efficiency of iterative algorithms for inverting frame operator on abstract Hilbert spaces. Recently, Kouchi and Rahimi [11] introduced Controlled frames in Hilbert C^*-modules.

In this paper we give the characterization of controlled g-frames and controlled dual g-frames in Hilbert C^*-modules and also we characterize controlled g-frames and get some comparable conditions for them. In the end we present the notion of controlled dual frames in Hilbert C^*-modules and give fundamental characterizations of these frames via operator machinery.

2. Preliminaries

Definition 2.1. [3]. Let $\mathcal{A}$ be a Banach algebra, an involution is a map $x \to x^*$ of $\mathcal{A}$ into itself such that for all x and y in $\mathcal{A}$ and all scalars α the following conditions hold:

(1) $(x^*)^* = x$.
(2) $(xy)^* = y^*x^*$.
(3) $(\alpha x + y)^* = \bar{\alpha}x^* + y^*$.

Definition 2.2. [3]. A C^*-algebra $\mathcal{A}$ is a Banach algebra with involution such that :

$$\|x^*x\| = \|x\|^2$$

for every x in $\mathcal{A}$.

Definition 2.3. [9]. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{U}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
(ii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$ is a norm on $\mathcal{H}$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$

Let Θ be a finite or countably index sets, N the set of natural numbers. For each $\xi \in \Theta$, we also reserve the notation $\text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_i)$ for the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ to $\mathcal{K}_\xi$ and $\text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is denoted by $\text{End}^*_{\mathcal{A}}(\mathcal{H})$. The set of all bounded linear operators on $\mathcal{H}$ with a bounded inverse is denoted by $GL(\mathcal{H})$ and $GL^+(\mathcal{H})$ be the set of all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse. We also denote

$$\bigoplus_{\xi\in\Theta} \mathcal{K}_\xi = \{\alpha = \{\alpha_\xi\} : \alpha_\xi \in \mathcal{K}_\xi \text{ and } \sum_{\xi\in\Theta} \langle \alpha_\xi, \alpha_\xi \rangle \text{ is norm convergent in } \mathcal{A} \}$$

.

Let $f = \{f_\xi\}_{\xi\in\Theta}$ and $g = \{g_\xi\}_{\xi\in\Theta}$, the inner product is defined by $\langle f, g \rangle = \sum_{\xi\in\Theta} \langle f_\xi, g_\xi \rangle$, we have $\bigoplus_{\xi\in\Theta} \mathcal{K}_\xi$ is a Hilbert A-module (see [12]).

Definition 2.4. [14] A sequence $\{\Upsilon_\xi \in \text{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{K}_i)\}_{i\in J}$ is called a g-frame in $\mathcal{H}$, if there exist two constants $A, B > 0$ such that for every $f \in \mathcal{H}$,

$$A\langle f,f\rangle \leq \sum_{i\in J}\langle \Upsilon_i f, \Upsilon_\xi f\rangle \leq B\langle f,f\rangle. \tag{2.1}$$

The numbers A and B are called the g-frames bounds of $\{\Upsilon_i \in \text{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{K}_i)\}_i$. If $A = B = \delta$, the g-frame is called δ-tight and if $A = B = 1$, it is called a Parseval g-frame. If only the right-hand inequality of (2.1) is satisfied, $\{\Upsilon_\xi\}_{\xi\in J}$ is called a g-Bessel sequence for $\mathcal{H}$.

Definition 2.5. Let $\{\Upsilon_\xi \in \text{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{K}_\xi)\}_\xi$ be a g-frames for $\mathcal{H}$ if

$$f = \sum_{\xi\in\Theta}\Phi^*_\xi\Upsilon_\xi f \text{ for } f \in \mathcal{H}$$

$\{\Phi_\xi\}_{\xi\in\Theta}$ is called an alternate dual g-frame for $\{\Upsilon_\xi\}_{\xi\in\Theta}$. Furthermore, $\{\Upsilon_\xi\}_{\xi\in\Theta}$ is an alternate dual g-frame for $\{\Phi_\xi\}_{\xi\in\Theta}$, that is to say

$$f = \sum_{\xi\in\Theta}\Upsilon^*_\xi\Phi_\xi f \text{ for } f \in \mathcal{H}.$$

Definition 2.6. Let $C_1, C_2 \in GL^+(\mathcal{H})$. A sequence of adjointable operators $\{\Upsilon_\xi\}_{\xi\in\Theta}$ is called a (C_1, C_2)-controlled g-frame for $\mathcal{H}$. If there exist two positive constants $A, B > 0$ such that

$$A\langle f,f\rangle \leq \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Upsilon_\xi C_2 f\rangle \leq B\langle f,f\rangle. \quad \forall f \in \mathcal{H} \tag{2.2}$$

the numbers A and B are called the lower and upper frame bounds for (C_1, C_2)-controlled g-frame, respectively.
If $\sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Upsilon_\xi C_2 f\rangle \leq B\langle f,f\rangle$ for all $f \in \mathcal{H}$, then $\{\Upsilon_\xi\}_{\xi\in\Theta}$ is called a (C_1, C_2)-controlled g-Bessel sequence for $\mathcal{H}$ If $C_2 = I_{\mathcal{H}}$, we call $\{\Upsilon_\xi\}_{\xi\in\Theta}$ a C_1-controlled g-frame for $\mathcal{H}$.

Lemma 2.7. [2] *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module, All positive and bounded operator $P : \mathcal{H} \to \mathcal{H}$ has a unique positive and bounded square root Q. We have*

(1) *P is self-adjoint $\Rightarrow$ Q is self-adjoint*
(2) *P is invertible $\Rightarrow$ Q is invertible*

Let $\{\Upsilon_\xi\}_{\xi\in J}$ be (C_1, C_2)-controlled g-Bessel sequence with bound B, the operator :

$$\mathcal{T}_{C_1\Upsilon C_2} : \bigoplus_{\xi\in\Theta}\mathcal{K}_\xi \to \mathcal{H}, \quad \mathcal{T}_{C_1\Upsilon C_2}\left(\{f_\xi\}_{\xi\in\Theta}\right) = \sum_{\xi\in\Theta}(C_1C_2)^{\frac{1}{2}}\Upsilon^*_\xi f_\xi, \quad \forall\{f_\xi\}_{\xi\in\Theta} \in \bigoplus_{\xi\in\Theta}\mathcal{K}_\xi$$

is called the synthesis operator of $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and

$$\mathcal{T}^*_{C_1\Upsilon C_2} : \mathcal{H} \to \bigoplus_{\xi\in\Theta}\mathcal{K}_\xi, \quad \mathcal{T}^*_{C_1\Upsilon C_2} f = \left\{\Upsilon_\xi(C_2C_1)^{\frac{1}{2}}f\right\}_{\xi\in\Theta}, \quad \forall f \in \mathcal{H}.$$

$\mathcal{T}^*_{C_1\Upsilon C_2}$ is called the analysis operator of $\{\Upsilon_\xi\}_{\xi\in\Theta}$.
When C_1, C_2 commute between them, and commute with the operator $\Upsilon^*_\xi\Upsilon_\xi$ for every ξ, the operator

$$S_{C_1\Upsilon C_2} : \mathcal{H} \to \mathcal{H}, \quad S_{C_1\Upsilon C_2} f = \sum_{\xi\in\Theta} C_2\Upsilon^*_\xi\Upsilon_\xi C_1 f, \quad \forall f \in \mathcal{H}$$

is called the frame operator of $\{\Upsilon_\xi\}_{\xi\in\Theta}$. We have, $S_{C_1\Upsilon C_2} = C_1 S_\Upsilon C_2$ is positive and invertible, where S_Υ is frame operator of g-frame $\{\Upsilon_\xi\}_{\xi\in\Theta}$, and is positive, invertible, bounded, self-adjoint, and $AI_\mathcal{H} \leq S_\Upsilon \leq BI_\mathcal{H}$.

For the above result one is referred to Hua and Huang [6]. therefore from now on we suppose that C_1, C_2 commute between them, and commute with the operator $\Upsilon^*_\xi\Upsilon_\xi$ for every ξ.

3. Controlled g-frames in Hilbert C^*- modules

Theorem 3.1. [11] *Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a linear operator. Then there exist two constants $0 < C_1 \leq C_2 < \infty$, such that $C_1 I_\mathcal{H} \leq \mathcal{T} \leq C_2 I_\mathcal{H}$ if and only if $\mathcal{T} \in GL^+(\mathcal{H})$*

Lemma 3.2. *Let $C_1, C_2 \in GL^+(\mathcal{H})$. Then the following assertions are equivalent:*

(1) *$\{\Upsilon_\xi\}_{\xi\in\Theta}$ is a (C_1, C_2)-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$*
(2) *$\{\Upsilon_\xi\}_{\xi\in\Theta}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$.*

Proof. (1) $\Rightarrow$ (2) Let$\{\Upsilon_\xi\}_{\xi\in\Theta}$ is a (C_1, C_2)-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$ with bounds A, B, and $h \in \mathcal{H}$, we have

$$\begin{aligned}
A\langle h, h\rangle &= A\langle (C_1C_2)^{\frac{1}{2}}(C_1C_2)^{-\frac{1}{2}}h, (C_1C_2)^{\frac{1}{2}}(C_1C_2)^{-\frac{1}{2}}h\rangle \\
&\leq A\left\|(C_1C_2)^{\frac{1}{2}}\right\|^2 \langle (C_1C_2)^{-\frac{1}{2}}h, (C_1C_2)^{-\frac{1}{2}}h\rangle \\
&\leq \left\|(C_1C_2)^{\frac{1}{2}}\right\|^2 \sum_{\xi\in\Theta}\left\langle \Upsilon_i C_1(C_1C_2)^{-\frac{1}{2}}h, \Upsilon_\xi C_2(C_1C_2)^{-\frac{1}{2}}h\right\rangle \\
&= \left\|(C_1C_2)^{\frac{1}{2}}\right\|^2 \left\langle C_2\sum_{\xi\in\Theta}\Upsilon_\xi C_1(C_1C_2)^{-\frac{1}{2}}h, \Upsilon_\xi(C_1C_2)^{-\frac{1}{2}}h\right\rangle \\
&= \left\|(C_1C_2)^{\frac{1}{2}}\right\|^2 \left\langle C_2 S_\Upsilon C_1(C_1C_2)^{-\frac{1}{2}}h, (C_1C_2)^{-\frac{1}{2}}h\right\rangle \\
&= \left\|(C_1C_2)^{\frac{1}{2}}\right\|^2 \langle S_\Upsilon h, h\rangle .
\end{aligned}$$

So

$$\frac{A}{\left\|(C_1C_2)^{\frac{1}{2}}\right\|^2}\langle h, h\rangle \leq \sum_{\xi\in\Theta}\langle \Upsilon_i h, \Upsilon_\xi h\rangle, \quad \forall h \in \mathcal{H}$$

For $h \in \mathcal{H}$,

$$\begin{aligned}\langle S_\Upsilon h, h\rangle &= \sum_{\xi\in\Theta} \langle \Upsilon_i h, \Upsilon_\xi h\rangle = \left\langle (C_1C_2)^{-\frac{1}{2}}(C_1C_2)^{\frac{1}{2}} S_\Upsilon h, h\right\rangle \\ &= \left\langle (C_1C_2)^{\frac{1}{2}} S_\Upsilon h, (C_1C_2)^{-\frac{1}{2}} h\right\rangle \\ &= \left\langle S_\Upsilon (C_1C_2)(C_1C_2)^{-\frac{1}{2}} h, (C_1C_2)^{-\frac{1}{2}} h\right\rangle \\ &= \left\langle C_1 S_\Upsilon C_2 (C_1C_2)^{-\frac{1}{2}} h, (C_1C_2)^{-\frac{1}{2}} h\right\rangle \\ &\leq B \left\| (C_1C_2)^{-\frac{1}{2}} \right\|^2 \langle h, h\rangle\end{aligned}$$

Which implies that $\{\Upsilon_\xi\}_{\xi\in\Theta}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$. (2) $\Rightarrow$ (1) Suppose that $\{\Upsilon_\xi\}_{\xi\in\Theta}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$ with bounds A_1, B_1. Then

$$\langle A_1 h, h\rangle \leq \langle S_\Upsilon h, h\rangle \leq \langle B_1 h, h\rangle \text{ for any } h \in \mathcal{H}.$$

Since $C_1, C_2 \in GL^+(\mathcal{H})$, by Lemma 3.1, there exist constants r, r_1, R, R_1 $(0 < r, r_1, R, R_1 < \infty)$ such that

$$rI_\mathcal{H} \leq C_1 \leq RI_\mathcal{H}, \quad r_1 I_\mathcal{H} \leq C_2 \leq R_1 I_\mathcal{H}.$$

Using $\langle C_1 S_\Upsilon h, h\rangle = \langle h, S_\Upsilon C_1 h\rangle = \langle h, C_1 S_\Upsilon h\rangle$, we get

$$rA \leq S_\Upsilon C_1 = C_1 S_\Upsilon \leq RB.$$

Identically, we have

$$rr_1 A \leq C_2 S_\Upsilon C_1 \leq RR_1 B.$$

Thus

$$rr_1 A\langle h, h\rangle \leq \sum_{\xi\in\Theta} \langle \Upsilon_\xi C_1 h, \Upsilon_\xi C_2 h\rangle \leq RR_1 B\langle h, h\rangle, \quad \forall h \in \mathcal{H}.$$

We conclude that $\{\Upsilon_i \in \mathrm{End}^*_\mathcal{A}(\mathcal{H}, \mathcal{K}_i)\}_{\xi\in J}$ is a (C_1, C_2)-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$. □

Lemma 3.3. *Let $C_1, C_2 \in GL^+(\mathcal{H})$. Then $\{\Upsilon_\xi \in \mathrm{End}^*_\mathcal{A}(\mathcal{H}, \mathcal{K}_\xi)\}_\xi$ is a (C_1, C_2)-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$ if and only if*

$$A\langle f, f\rangle \leq \sum_{\xi\in\Theta} \left\langle \Upsilon_\xi (C_2C_1)^{\frac{1}{2}} f, \Upsilon_\xi (C_2C_1)^{\frac{1}{2}} f\right\rangle \leq B\langle f, f\rangle, \quad \forall f \in \mathcal{H},$$

Proof. Let $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Upsilon_\xi C_2 f\rangle &= \left\langle \sum_{\xi\in\Theta} C_2\Upsilon_\xi^*\Upsilon_\xi C_1 f, f\right\rangle = \langle C_2 S_\Upsilon C_1 f, f\rangle\\ &= \langle C_2 C_1 S_\Upsilon f, f\rangle = \left\langle (C_2C_1)^{\frac{1}{2}} S_\Upsilon (C_2C_1)^{\frac{1}{2}} f, f\right\rangle\\ &= \left\langle \sum_{\xi\in\Theta} (C_2C_1)^{\frac{1}{2}}\Upsilon_\xi^*\Upsilon_\xi (C_2C_1)^{\frac{1}{2}} f, f\right\rangle\\ &= \sum_{\xi\in\Theta}\left\langle \Upsilon_\xi (C_2C_1)^{\frac{1}{2}} f, \Upsilon_\xi (C_2C_1)^{\frac{1}{2}} f\right\rangle\end{aligned}$$

consequently, $\{\Upsilon_\xi : \xi\in\Theta\}$ is a (C_1, C_2)-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$ is equivalent to

$$A\langle f, f\rangle \leq \sum_{\xi\in\Theta}\left\langle \Upsilon_\xi (C_2C_1)^{\frac{1}{2}} f, \Upsilon_\xi (C_2C_1)^{\frac{1}{2}} f\right\rangle \leq B\langle f, f\rangle, \quad \forall f\in\mathcal{H},$$

Thus $\{\Upsilon_\xi : \xi\in\Theta\}$ is a $\left((C_2C_1)^{\frac{1}{2}}, (C_2C_1)^{\frac{1}{2}}\right)$-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$.

□

Lemma 3.4. *Let $C_1, C_2 \in GL^+(\mathcal{H})$. Then the following statements are equivalent:*

(1) $\{\Upsilon_\xi\}_{\xi\in\Theta}$ *is a (C_1, C_2)-controlled g-frame for $\mathcal{H}$ with respect to* $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$

(2) $\{\Upsilon_\xi\}_{\xi\in\Theta}$ *is a C_2C_1-controlled g-frame for $\mathcal{H}$ with respect to* $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$.

Proof. Let $C_1, C_2 \in GL^+(\mathcal{H})$,for $f\in\mathcal{H}$, we have

$$\begin{aligned}\sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Upsilon_\xi C_2 f\rangle &= \left\langle \sum_{\xi\in\Theta} C_2\Upsilon_\xi^*\Upsilon_\xi C_1 f, f\right\rangle\\ &= \langle C_2 S_\Upsilon C_1 f, f\rangle\\ &= \langle C_2 C_1 S_\Upsilon f, f\rangle\\ &= \left\langle \sum_{\xi\in\Theta} (C_2C_1)\Upsilon_\xi^*\Upsilon_\xi f, f\right\rangle\\ &= \sum_{\xi\in\Theta}\langle \Upsilon_\xi (C_2C_1) f, \Upsilon_\xi f\rangle\end{aligned}$$

and we have

$$A\langle f, f\rangle \leq \sum_{\xi\in\Theta}\langle \Upsilon_\xi (C_2C_1) f, f\rangle \leq B\langle f, f\rangle, \quad \forall f\in\mathcal{H},$$

Hence, $\{\Upsilon_\xi\}_{\xi\in\Theta}$ is a C_2C_1-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$ □

Lemma 3.5. *Let $C_1, C_2 \in GL^+(\mathcal{H})$. Then the following statements are equivalent:*

(1) $\{\Upsilon_\xi\}_{\xi\in\Theta}$ *is a (C_1, C_2)-controlled g-frame for $\mathcal{H}$ with respect to* $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$

(2) $\{v_{\xi,k}\}_{\xi\in\Theta, k\in K_\xi}$ *is a (C_1, C_2)-controlled for $\mathcal{H}$, where $v_{\xi,k} = \Upsilon_\xi^* e_{\xi,k}$, for $\xi\in\Theta$ and $k \in K_\xi$*

Proof. Let $\{e_{\xi,k}\}_{k\in K_\xi}$ be an orthonormal basis for $\mathcal{K}_\xi$ for each $\xi \in \Theta$, consequently for any $h \in \mathcal{H}$, we have $\Upsilon_\xi h \in \mathcal{K}_\xi$. It follows that

$$\Upsilon_\xi C_1 h = \sum_{k\in K_\xi} \langle \Upsilon_\xi C_1 h, e_{\xi,k}\rangle e_{\xi,k} = \sum_{k\in K_\xi} \left\langle h, C_1 \Upsilon_\xi^* e_{\xi,k}\right\rangle e_{\xi,k}.$$

and

$$\Upsilon_\xi C_2 h = \sum_{k\in K_\xi} \langle \Upsilon_\xi C_2 h, e_{\xi,k}\rangle e_{\xi,k} = \sum_{k\in K_\xi} \left\langle h, C_2 \Upsilon_\xi^* e_{\xi,k}\right\rangle e_{\xi,k}.$$

We have

$$\langle \Upsilon_\xi C_1 h, \Upsilon_\xi C_2 h\rangle = \sum_{k\in K_\xi} \left\langle h, C_1 \Upsilon_\xi^* e_{\xi,k}\right\rangle \left\langle C_2 \Upsilon_\xi^* e_{\xi,K}, h\right\rangle = \sum_{k\in K_\xi} \langle h, C_1 v_{\xi,k}\rangle \langle C_2 v_{\xi,k}, h\rangle.$$

Hence

$$A\langle h,h\rangle \le \sum_{\xi\in\Theta} \langle \Upsilon_\xi C_1 h, \Upsilon_\xi C_2 h\rangle = \sum_{\xi\in\Theta}\sum_{k\in K_\xi} \langle h, C_1 v_{\xi,k}\rangle \langle C_2 v_{\xi,k}, h\rangle \le B\langle h,h\rangle$$

is equivalent to

$$A\langle h,h\rangle \le \sum_{\xi\in\Theta}\sum_{k\in K_\xi} \langle h, C_1 v_{\xi,k}\rangle \langle C_2 v_{\xi,k}, h\rangle \le B\langle h,h\rangle \text{ for any } h \in \mathcal{H}.$$

□

Lemma 3.6. *Let $C_1, C_2 \in GL^+(\mathcal{H})$. Then the following assertions are equivalent:*

(1) $\{\Upsilon_\xi\}_{\xi\in\Theta}$ *is a (C_1, C_2)-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$.*

(2) $\{C_1 v_{\xi,k}\}_{\xi\in\Theta, k\in K_\xi}$ *is a $C_2C_1^{-1}$-controlled for $\mathcal{H}$, where $v_{\xi,k} = \Upsilon_\xi^* e_{\xi,k}$, for $\xi \in \Theta$ and $k \in K_\xi$.*

Proof. By the proof of Lemma 3.5, we have

$$\sum_{\xi\in\Theta} \langle \Upsilon_\xi C_1 f, \Upsilon_\xi C_2 f\rangle = \sum_{\xi\in\Theta}\sum_{k\in k_\xi} \left\langle f, C_1 \Upsilon_\xi^* e_{\xi,k}\right\rangle \left\langle C_2 \Upsilon_\xi^* e_{\xi,K}, f\right\rangle.$$

Let's put $f_{\xi,k} = C_1 v_{\xi,k}, v_{\xi,k} = \Upsilon_\xi^* e_{\xi,k}$, so

$$A\langle f,f\rangle \le \sum_{\xi\in\Theta} \langle \Upsilon_\xi C_1 f, \Upsilon_\xi C_2 f\rangle \le B\langle f,f\rangle$$

is equivalent to

$$A\langle f,f\rangle \le \sum_{\xi\in\Theta}\sum_{k\in K_\xi} \langle f, f_{\xi,k}\rangle \left\langle C_2 C_1^{-1} v_{\xi,k}, f\right\rangle \le B\langle f,f\rangle \text{ for any } f \in \mathcal{H}.$$

□

4. Controlled dual g-frames in Hilbert C^*-modules

Definition 4.1. Let $C_1, C_2 \in GL^+(\mathcal{H})$, $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and $\{\Phi_\xi\}_{\xi\in\Theta}$ be (C_1, C_1)-controlled and (C_2, C_2)-controlled g-Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$, respectively. If for every $h \in \mathcal{H}$,

$$h = \sum_{\xi\in\Theta} C_1 \Upsilon_\xi^* \Phi_\xi C_2 h$$

Then $\{\Phi_\xi\}_{\xi\in\Theta}$ is called a (C_1, C_2)-controlled dual g-frame of $\{\Upsilon_\xi\}_{\xi\in\Theta}$.

Definition 4.2. Let $C_1, C_2 \in GL^+(\mathcal{H})$, $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and $\{\Phi_\xi\}_{\xi\in\Theta}$ be (C_1, C_1)-controlled and (C_2, C_2)-controlled g Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$, respectively. if for any $h \in \mathcal{H}$,

$$S_{C_1\Upsilon\Phi C_2} h = \sum_{\xi\in\Theta} C_1 \Upsilon_\xi^* \Phi_\xi C_2 h.$$

$S_{C_1\Upsilon\Phi C_2}$ is called a (C_1, C_2)-controlled dual g-frame operator for this pair of controlled g-Bessel sequence.

We clearly see that $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and $\{\Phi_\xi\}_{\xi\in\Theta}$ are also two g-Bessel sequences. $S_{C_1\Upsilon\Phi C_2}$ is a well-defined and bounded , and we have

$$S_{C_1\Upsilon\Phi C_2} = \mathcal{T}_{C_1\Upsilon C_1} \mathcal{T}_{C_2\Phi C_2}^* = C_1 \mathcal{T}_\Upsilon \mathcal{T}_\Phi^* C_2 = C_1 S_{\Upsilon\Phi} C_2,$$

Proposition 4.3. *Let $C_1, C_2 \in GL^+(\mathcal{H})$, $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and $\{\Phi_\xi\}_{\xi\in\Theta}$ be (C_1, C_1)-controlled and (C_2, C_2)-controlled g-Bessel sequences with bounds B_Υ and B_Φ, respectively.*
If $S_{C_1\Upsilon\Phi C_2}$ is bounded below, then $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and $\{\Phi_\xi\}_{\xi\in\Theta}$ are (C_1, C_1)-controlled and (C_2, C_2)-controlled g-frames, respectively.

Proof. Let us assume there is a constant $\lambda > 0$ such that

$$\|S_{C_1\Upsilon\Phi C_2} f\| \geq \lambda \langle f, f\rangle \text{ for all } f \in \mathcal{H}.$$

By Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
\lambda \|\langle f, f\rangle\|^{\frac{1}{2}} \leq \|S_{C_1\Upsilon\Phi C_2} f\| &= \sup_{\|g\|=1} \left\| \left\langle \sum_{\xi\in\Theta} C_1 \Upsilon_\xi^* \Phi_\xi C_2 f, g \right\rangle \right\| \\
&= \sup_{\|g\|=1} \left\| \sum_{\xi\in\Theta} \langle \Phi_\xi C_2 f, \Upsilon_\xi C_1 g\rangle \right\| \\
&\leq \sup_{\|g\|=1} \sqrt{\left\| \sum_{\xi\in\Theta} \langle \Phi_\xi C_2 f, \Phi_\xi C_2 f\rangle \right\|} \sqrt{\left\| \sum_{\xi\in\Theta} \langle \Upsilon_\xi C_1 g, \Upsilon_\xi C_1 g\rangle \right\|} \\
&\leq \sqrt{B_\Upsilon} \sqrt{\left\| \sum_{\xi\in\Theta} \langle \Phi_\xi C_2 f, \Phi_\xi C_2 f\rangle \right\|}.
\end{aligned}$$

Thus

$$\frac{\lambda^2}{B_\Upsilon}\langle f, f\rangle \leq \sum_{\xi\in\Theta}\langle \Phi_\xi C_2 f, \Phi_\xi C_2 f\rangle \text{ for } f \in \mathcal{H}$$

On the other hand, Since

$$S^*_{C_1\Upsilon\Phi C_2} = (C_1 S_{\Upsilon\Phi} C_2)^* = C_2 S^*_{\Upsilon\Phi} C_1 = C_2 S_{\Phi\Upsilon} C_1 = S_{C_2\Phi\Upsilon C_1},$$

then $S_{C_2\Phi\wedge C_1}$ is also bound below. Similarly, we can prove that

$$\frac{\lambda^2}{B_\Phi}\langle f, f\rangle \leq \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_2 f, \Upsilon_\xi C_2 f\rangle \text{ for } f \in \mathcal{H}$$

Hence we conclude that $\{\Upsilon_\xi\}_{\xi\in\Theta}$ is a (C_1, C_1)-controlled g-frames. □

Theorem 4.4. *Let $C_1, C_2 \in GL^+(\mathcal{H})$, $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and $\{\Phi_\xi\}_{\xi\in\Theta}$ be (C_1, C_1)-controlled and (C_2, C_2)-controlled g Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$, respectively. Then the following statements are equivalent:*

(1) $f = \sum_{\xi\in\Theta} C_1\Upsilon^*_\xi\Phi_\xi C_2 f, \forall f \in \mathcal{H}$.
(2) $f = \sum_{\xi\in\Theta} C_2\Phi^*_\xi\Upsilon_\xi C_1 f, \forall f \in \mathcal{H}$.
(3) $\langle f, g\rangle = \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 g\rangle = \sum_{\xi\in\Theta}\langle \Phi_\xi C_2 f, \Upsilon_\xi C_1 g\rangle, \forall f, g \in \mathcal{H}$.
(4) $\langle f, f\rangle = \sum_{\xi\in\Theta}^{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 f\rangle = \sum_{\xi\in\Theta}^{\xi\in\Theta}\langle \Phi_\xi C_2 f, \Upsilon_\xi C_1 f\rangle, \forall f \in \mathcal{H}$.

In case the equivalent conditions are satisfied, $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and $\{\Phi_\xi\}_{\xi\in\Theta}$ are (C_1, C_1)-controlled and (C_2, C_2) controlled g-frames, respectively.

Proof. (1) $\Leftrightarrow$ (2). Let $\mathcal{T}_{C_1\Upsilon C_1}$ and $\mathcal{T}_{C_1\Phi C_1}$ be the synthesis operator of the (C_1, C_1)-controlled g-Bessel sequence $\{\Upsilon_\xi\}_{\xi\in\Theta}$ and (C_2, C_2)-controlled g-Bessel sequence $\{\Phi_\xi\}_{\xi\in\Theta}$ respectively. Moreover, we see that $\mathcal{T}_{C_1AC_1}\mathcal{T}^*_{C_2\Phi C_2} = I_\mathcal{H}$, wich is equivalent to $\mathcal{T}_{C_2\Phi C_2}\mathcal{T}^*_{C_1AC_1} = I_\mathcal{H}$, which is identical to the statement (2). Conversely, (2) implies (1) similarly.
(2) $\Rightarrow$ (3). suppose that for any $f, g \in \mathcal{H}$ we have $f = \sum_{\xi\in\Theta} C_2\Phi^*_\xi\Upsilon_\xi C_1 f$, then

$$\langle f, g\rangle = \langle\sum_{\xi\in\Theta} C_2\Phi^*_\xi\Upsilon_\xi C_1 f, g\rangle = \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 g\rangle = \sum_{\xi\in\Theta}\langle \Phi_\xi C_2 f, \Upsilon_\xi C_1 g\rangle$$

(2) $\Leftarrow$ (3). suppose that for any $f, g \in \mathcal{H}, \langle f, g\rangle = \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 g\rangle$ shows that

$$\left\langle f - \sum_{\xi\in\Theta} C_2\Phi^*_\xi\Upsilon_\xi C_1 f, g\right\rangle = 0, \quad \forall g \in \mathcal{H}$$

Hence (2) is followed.
(3) $\Rightarrow$ (4) is evident .

(4) $\Rightarrow$ (3). using condition (4), we have

$$\begin{aligned}\langle f+g, f+g\rangle &= \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1(f+g), \Phi_\xi C_2(f+g)\rangle \\ &= \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f + \Upsilon_\xi C_1 g, \Phi_\xi C_2 f + \Phi_\xi C_2 g\rangle \\ &= \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 f\rangle + \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 g\rangle \\ &\quad + \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 g, \Phi_\xi C_2 f\rangle + \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 g, \Phi_\xi C_2 g\rangle .\end{aligned}$$

Also,

$$\begin{aligned}\langle f-g, f-g\rangle = \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Gamma_\xi C_2 f\rangle - \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 g\rangle \\ - \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 g, \Phi_\xi C_2 f\rangle + \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 g, \Phi_\xi C_2 g\rangle .\end{aligned}$$

$$\begin{aligned}\langle f+\mathrm{i}g, f+\mathrm{i}g\rangle = \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 f\rangle - \mathrm{i}\sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 g\rangle \\ + \mathrm{i}\sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 g, \Phi_\xi C_2 f\rangle + \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 g, \Phi_\xi C_2 g\rangle .\end{aligned}$$

$$\begin{aligned}\langle f-\mathrm{i}g, f-\mathrm{i}g\rangle = \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 f\rangle + \mathrm{i}\sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 g\rangle \\ -\mathrm{i}\sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 g, \Phi_\xi C_2 f\rangle + \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 g, \Phi_\xi C_2 g\rangle .\end{aligned}$$

and from the polarization identity,

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{4}\left(\langle f+g, f+g\rangle - \langle f-g, f-g\rangle + \mathrm{i}\langle f+\mathrm{i}g, f+\mathrm{i}g\rangle - \mathrm{i}\langle f-\mathrm{i}g, f-\mathrm{i}g\rangle\right) \\ &= \sum_{\xi\in\Theta}\langle \Upsilon_\xi C_1 f, \Phi_\xi C_2 g\rangle .\end{aligned}$$

□

Lemma 4.5. [15] *Let $C_1, C_2 \in GL^+(\mathcal{H})$, the operator*

$$\mathcal{T}_{C_1\Upsilon C_2} : \bigoplus_{\xi\in\Theta}\mathcal{K}_\xi \to \mathcal{H}, \quad \mathcal{T}_{C_1\Upsilon C_2}\left(\{f_\xi\}_{\xi\in\Theta}\right) = \sum_{\xi\in\Theta}\sqrt{C_1C_2}\,\Upsilon_\xi^* f_\xi$$

is well-defined and bounded with $\|\mathcal{T}_{C_1\Upsilon C_2}\| \le \sqrt{B}$. If and only if $\{\Upsilon_\xi : \xi\in\Theta\}$ is (C_1, C_2)-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$ with bound B.

Theorem 4.6. *Let $C_1, C_2 \in GL^+(\mathcal{H})$. A sequence $\{\Upsilon_\xi : \xi\in\Theta\} \subset \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_\xi)$ be a (C_1, C_1)-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$. Then the following statements are equivalent:*

1. *a (C_2, C_2)-controlled g-frame $\{\Phi_\xi\}_{\xi\in\Theta}$ is a (C_1, C_2)-controlled dual g-frame of $\{\Upsilon_\xi\}_{\xi\in\Theta}$*
2. *$C_2\Phi_\xi^* e_{\xi,k} = \mathcal{U}(e_{\xi,k}\delta_\xi)$, $\quad \xi\in\Theta, k\in K_\xi\subset\mathbb{Z}$, where $\mathcal{U} : \bigoplus_{\xi\in\Theta}\mathcal{K}_\xi \to \mathcal{H}$ is a bounded left-inverse of $\mathcal{T}^*_{C_1\wedge C_1}$.*

Proof. We suppose that $\{k_\xi\}_{\xi\in\Theta} \in \bigoplus_{\xi\in\Theta} \mathcal{K}_\xi$, thus

$$\{k_\xi\}_{\xi\in\Theta} = \sum_{\xi\in\Theta} k_\xi \delta_\xi = \sum_{\xi\in\Theta}\sum_{k\in K_\xi} \langle k_\xi, e_{\xi,k}\rangle e_{\xi,k}\delta_\xi. \tag{4.1}$$

where δ is the Kronecker symbol.

We have $\{e_{\xi,k}\delta_\xi\}_{\xi\in\Theta,k\in K_\xi}$ is an orthonormal basis of $\bigoplus_{\xi\in\Theta} \mathcal{K}_\xi$. If there exist $\mathcal{U} : \bigoplus_{\xi\in\Theta} \mathcal{K}_\xi \to \mathcal{H}$ is a bounded left-inverse of $\mathcal{T}^*_{C_1\Upsilon C_1}$ such that

$$\mathcal{U}(e_{\xi,k}\delta_\xi) = C_2\Phi^*_\xi e_{\xi,k}, \quad \xi \in \Theta, k \in K_\xi.$$

applying Lemma 4.1,$\{\Phi_\xi\}_{\xi\in\Theta}$ is a (C_2, C_2)-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$. For every $h \in \mathcal{H}$, the equation (4,1) gives us

$$\begin{aligned} h = \mathcal{U}\mathcal{T}^*_{C_1\Upsilon C_1}h &= \mathcal{U}\left(\sum_{\xi\in\Theta}\sum_{k\in K_\xi} \langle \Upsilon_\xi C_1 h, e_{\xi,k}\rangle e_{\xi,k}\delta_\xi\right) \\ &= \sum_{\xi\in\Theta}\sum_{k\in K_\xi} \left\langle h, C_1\Upsilon^*_\xi e_{\xi,k}\right\rangle \mathcal{U}(e_{\xi,k}\delta_\xi) \\ &= \sum_{\xi\in\Theta} C_2\Phi^*_\xi \sum_{k\in K_\xi} \langle h, C_1 v_{\xi,k}\rangle e_{\xi,k} \\ &= \sum_{\xi\in\Theta} C_2\Phi^*_\xi\Upsilon_\xi C_1 h, \end{aligned}$$

where $v_{\xi,k} = \Upsilon^*_\xi e_{\xi,k}$. we have, $\{\Phi_\xi\}_{\xi\in\Theta}$ is a (C_1, C_2)-controlled dual g-frame of $\{\Upsilon_\xi\}_{\xi\in\Theta}$

Conversely, For $h \in \mathcal{H}$, we have

$$h = \sum_{\xi\in\Theta} C_1\Upsilon^*_\xi\Phi_\xi C_2 h = \sum_{\xi\in\Theta} C_2\Phi^*_\xi\Upsilon_\xi C_1 h$$

which is $\mathcal{T}_{C_2\Phi C_2}\mathcal{T}^*_{C_1\Upsilon C_1} = I_\mathcal{H}$. Let $\mathcal{U} = \mathcal{T}_{C_2\Phi C_2}$, then $\mathcal{U} : \bigoplus_{\xi\in\Theta} \mathcal{K}_\xi \to \mathcal{H}$ is a bounded left-inverse of $\mathcal{T}^*_{C_1\Upsilon C_1}$. Then

$$h = \sum_{\xi\in\Theta}\sum_{k\in K_\xi} \langle h, C_1 v_{\xi,k}\rangle C_2\Phi^*_\xi e_{\xi,k} = \sum_{\xi\in\Theta}\sum_{k\in K_\xi} \langle h, C_1 v_{\xi,k}\rangle \mathcal{U}(e_{\xi,k}\delta_\xi), \quad \forall h \in \mathcal{H}$$

since $\{e_{\xi,k}\}_{k\in K_\xi}$ is an orthonormal basis of $\mathcal{K}_\xi$, we have

$$C_2\Phi^*_\xi e_{\xi,k} = \mathcal{U}(e_{\xi,k}\delta_\xi), \quad \xi \in \Theta, k \in K_\xi.$$

□

Theorem 4.7. *Let $\Delta \in GL^+(\mathcal{H})$, $\{\Upsilon_\xi\}_{\xi\in\Theta}$ be a (Δ, Δ)-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$ with the frame operator and synthesis operator $S_{\Delta\Upsilon\Delta}$ and $\mathcal{T}_{\Delta\Upsilon\Delta}$, respectively. Then A sequence $\{\Phi_\xi : \xi \in \Theta\} \subset \mathrm{End}^*_\mathcal{A}(\mathcal{H}, \mathcal{K}_\xi)$ is a Δ-controlled dual g-frame of $\{\Upsilon_\xi\}_{\xi\in\Theta}$ if and only if*

$$\Phi_\xi h = (\mathcal{T}h)_\xi + \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h, \quad \xi \in \Theta, h \in \mathcal{H}$$

where $\mathcal{T} : \mathcal{H} \to \bigoplus_{\xi\in\Theta} \mathcal{K}_\xi$ is a bounded linear operator satisfying $\mathcal{T}_{\Delta\Upsilon\Delta}\mathcal{T} = 0$

Proof. If $\mathcal{T} : \mathcal{H} \to \bigoplus_{\xi\in\Theta} \mathcal{K}_\xi$ is a bounded linear operator satisfying $\mathcal{T}_{\Delta\Upsilon\Delta}\mathcal{T} = 0$. Then $\{\Phi_\xi : \xi \in \Theta\} \subset \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_\xi)$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Theta}$. In fact, for any $h \in \mathcal{H}$ we have

$$\begin{aligned}
\sum_{\xi\in\Theta}\langle \Phi_\xi h, \Phi_\xi h\rangle &= \sum_{\xi\in\Theta}\langle (\mathcal{T}h)_\xi + \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h, (\mathcal{T}h)_\xi + \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h\rangle \\
&\le 2\left(\sum_{\xi\in\Theta}\langle \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}Pf, \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h\rangle + \|\mathcal{T}h\|^2\right) \\
&\le 2\left(\sum_{\xi\in\Theta}\langle \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta, \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta\rangle\langle h, h\rangle + \|\mathcal{T}\|^2\langle h, h\rangle\right) \\
&\le 2\left(B\left\|S^{-1}_{\Delta\Upsilon\Delta}\Delta\right\|^2 + \|\mathcal{T}\|^2\right)\langle h, h\rangle,
\end{aligned}$$

where B is the upper bound of $\{\Upsilon_\xi\}_{\xi\in\Theta}$. Furthermore,

$$\begin{aligned}
\sum_{\xi\in\Theta}\Delta\Upsilon^*_\xi\Phi_\xi h &= \sum_{\xi\in\Theta}\Delta\Upsilon^*_\xi\left((\mathcal{T}h)_\xi + \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h\right) \\
&= \mathcal{T}_{\Delta\Upsilon\Delta}\mathcal{T}h + \sum_{\xi\in\Theta}\Delta\Upsilon^*_\xi\Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h \\
&= 0 + S^{-1}_{\Delta\Upsilon\Delta}\sum_{\xi\in\Theta}\Delta\Upsilon^*_\xi\Upsilon_\xi\Delta h \\
&= h.
\end{aligned}$$

Thus $\{\Phi_\xi : \xi \in \Theta\} \subset \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_\xi)$ is a Δ-controlled dual g-frame of $\{\Upsilon_\xi\}_{\xi\in\Theta}$. On the other hand. Suppose that $\{\Phi_\xi \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_\xi) : \xi \in \Theta\}$ is a Δ-controlled dual g-frame of $\{\Upsilon_\xi\}_{\xi\in\Theta}$. Now we consider the operator $\mathcal{T}$ which is defined by

$$\mathcal{T} : \mathcal{H} \to \bigoplus_{\xi\in\Theta}\mathcal{K}_\xi, \quad h \mapsto Sh \quad (\forall h \in \mathcal{H})$$

satisfying

$$\Phi_\xi h = (\mathcal{T}h)_\xi + \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h, \quad \xi \in \Theta.$$

Hence

$$\begin{aligned}
\|\mathcal{T}h\|^2 &= \sum_{\xi\in\Theta}\langle \Phi_\xi h - \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h, \Phi_\xi h - \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h\rangle \\
&\le \sum_{\xi\in\Theta}\langle \Phi_\xi h, \Phi_\xi h\rangle + \sum_{\xi\in\Theta}\langle \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h, \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h\rangle \\
&\quad + 2\left(\sum_{\xi\in\Theta}\langle \Phi_\xi h, \Phi_\xi h\rangle\right)^{\frac{1}{2}}\left(\sum_{\xi\in\Theta}\langle \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h, \Upsilon_\xi S^{-1}_{\Delta\Upsilon\Delta}\Delta h\rangle\right)^{\frac{1}{2}} \\
&\le \left(C + D^{-1} + 2\sqrt{CD^{-1}}\right)\langle h, h\rangle,
\end{aligned}$$

then $\mathcal{T}$ is a linear bounded operator. Furthermore, for any $h, g \in \mathcal{H}$, we have

$$\begin{aligned}\langle \mathcal{T}_{\Delta\Upsilon\Delta}\mathcal{T}h, g\rangle = \sum_{\xi\in\Theta}\left\langle \Delta\Upsilon_\xi^*\mathcal{T}h, g\right\rangle &= \sum_{\xi\in\Theta}\left\langle \Delta\Upsilon_\xi^*\left(\Phi_\xi h - \Upsilon_\xi S_{\Delta\Upsilon\Delta}^{-1}\Delta h\right), g\right\rangle \\ &= \sum_{\xi\in\Theta}\left\langle \Delta\Upsilon_\xi^*\Phi_\xi h, g\right\rangle - \sum_{\xi\in\Theta}\left\langle \Delta\Upsilon_\xi^*\Upsilon_\xi S_{\Delta\Upsilon\Delta}^{-1}\Delta h, g\right\rangle \\ &= \langle h, g\rangle - \langle h, g\rangle = 0.\end{aligned}$$

Hence we conclude that $\mathcal{T}_{\Delta\Upsilon\Delta}\mathcal{T} = 0$. □

Declarations

Availablity of data and materials

Not applicable.

Competing interest

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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