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Woven generalized fusion frame in Hilbert C^*-module

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Abstract

Woven frames have been introduced for studying some problems arising in distributed signal processing. Because of some potential applications such as in wireless sensor networks and pre-processing of signals. In this paper, we introduced the notion of a woven $g-$fusion frame in Hilbert C^*-modules, we gives some properties and we study perturbation of weaving $g-$fusion frames.

Keywords: Fusion frame, g-fusion frame, woven g-fusion frame, C^*-algebras, Hilbert C^*-modules.

1. Introduction Basis is one of the most important concepts in Vector Spaces study. However, Frames generalise orthonormal bases and were introduced by Duffin and Schaefer [6] in 1952 to analyse some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [9] for signal processing. In 2000, Frank-larson [8] introduced the concept of frames in Hilbet C^*-modules as a generalization of frames in Hilbert spaces. The basic idea was to consider modules over C^*-algebras of linear spaces and to allow the inner product to take values in the C^*-algebras [14]. A. Khosravi and B. Khosravi [13] introduced the fusion frames and $g-$frame theory in Hilbert C^*-modules. Afterwards, A. Alijani and M. Dehghan consider frames with

C^*-valued bounds [2] in Hilbert C^*-modules. N. Bounader and S. Kabbaj [5] and A. Alijani [1] introduced the $*$-g-frames which are generalizations of g-frames in Hilbert C^*-modules. In 2016, Z. Xiang and Y. Li [23] give a generalization of $g-$frames for operators in Hilbert C^*-modules. Recently, Fakhr-dine Nhari et al. [15] introduced the concepts of g-fusion frame and K-g-fusion frame in Hilbert C^*-modules. Bemrose et al. [4] introduced a new concept of weaving frames in separable Hilbert spaces. This notion has potential applications in distributed signal processing and wireless sensor networks. Weaving Frames in Hilbert C^*-Modules introduced by X. Zhao and P. Li [22]. For more on frame in Hilbert C^*-modules see [11, 17, 18, 19, 20, 21] and references therein. In this paper, we introduced the notion of a woven $g-$fusion frame in Hilbert C^*-modules, we gives some properties and we study perturbation of weaving $g-$fusion frames.

The paper is organized as follows, we continue this introductory section we briefly recall the definitions and basic properties of Hilbert C^*-modules. In section 2, we introduce the concept of woven $g-$fusion frames by extending and improving the notion of $g-$fusion frames and weaving frames. We investigate the structure of woven $g-$fusion frames and characterize them. We start the section 3 with Paley-Wiener perturbation of weaving $g-$fusion frames and continue two results of perturbations in the sequel.

Throughout this paper, H is considered to be a countably generated Hilbert $\mathcal{A}-$module. Let $\{H_i\}_{i\in I}$ be a collection of Hilbert $\mathcal{A}-$module and $\{W_i\}_{i\in I}$ be a collection of closed orthogonally complemented submodules of H, where I be finite or countable index set. $End^*_{\mathcal{A}}(H,H_i)$ is the set of all adjointable operator from H to H_i. In particular $End^*_{\mathcal{A}}(H)$ denote the set of all adjointable operators on H. P_{W_i} denote the orthogonal projection onto the closed submodule orthogonally complemented W_i of H. Define the module

$$l^2(\{H_i\}_{i\in I}) = \{\{x_i\}_{i\in I} : x_i \in H_i, \|\sum_{i\in I}\langle x_i, x_i\rangle\| < \infty\}$$

with $\mathcal{A}-$valued inner product $\langle x, y\rangle = \sum_{i\in I}\langle x_i, y_i\rangle$, where $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$, clearly $l^2(\{H_i\}_{i\in I})$ is a Hilbert $\mathcal{A}-$module.

In the following we briefly recall the definitions and basic properties of Hilbert $\mathcal{A}$-modules.

Definition 0.1. [12]. Let $\mathcal{A}$ be a unital C^*-algebra and H be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and U are compatible. H is a pre-Hilbert $\mathcal{A}$-module if H is equipped with an $\mathcal{A}$-valued inner product $\langle.,.\rangle : H \times H \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x\rangle \geq 0$ for all $x \in H$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in H$.

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in H$.

For $x \in H$, we define $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$. If H is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on H is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}$ for $x \in H$.

Lemma 0.2. [3]. *Let H and K two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(H, K)$. Then the following statements are equivalent:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in K$.*

(iii) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle$ for all $x \in K$.*

Lemma 0.3. [2]. *Let U and H two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(U, H)$. Then:*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 0.4. [7] *Let E, H and K be Hilbert $\mathcal{A}-$modules, $T \in End^*_{\mathcal{A}}(E, K)$ and $T' \in End^*_{\mathcal{A}}(H, K)$. Then the following two statements are equivalent:*

(1) *$T'(T')^* \leq \lambda TT^*$ for some $\lambda > 0$;*

(2) *There exists $\mu > 0$ such that $\|(T')^*z\| \leq \mu\|T^*z\|$ for all $z \in K$.*

Definition 0.5. [15] Let $\{W_i\}_{i\in I}$ be a sequence of closed orthogonally complemented submodules of H, $\{v_i\}_{i\in I}$ be a familly of positive weights in $\mathcal{A}$, i.e., each v_i is a positive invertible element from the center of the C^*-algebra $\mathcal{A}$ and $\Lambda_i \in End^*_{\mathcal{A}}(H, H_i)$ for all

$i \in I$. We say that $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $g-$fusion frame for H if and only if there exists two constants $0 < A \leq B < \infty$ such that

$$A\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H. \tag{0.1}$$

The constants A and B are called the lower and upper bounds of $g-$fusion frame, respectively. If $A = B$ then Λ is called tight g-fusion frame and if $A = B = 1$ then we say Λ is a Parseval $g-$fusion frame. If Λ satisfies the inequality

$$\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H.$$

then it is called a $g-$fusion bessel sequence with bound B in H.

Definition 0.6. [15] let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion bessel sequence for H. Then the operator $T_\Lambda : l^2(\{H_i\}_{i\in I}) \to H$ defined by

$$T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} v_i P_{W_i} \Lambda_i^* f_i, \qquad \forall \{f_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I}).$$

Is called synthesis operator. We say the adjoint U_Λ of the synthesis operator the analysis operator and it is defined by $U_\Lambda : \mathcal{H} \to l^2(\{H_i\}_{i\in I})$ such that

$$U_\Lambda(f) = \{v_i \Lambda_i P_{W_i}(f)\}_{i\in I}, \qquad \forall f \in H.$$

The operator $S_\Lambda : H \to H$ defined by

$$S_\Lambda f = T_\Lambda U_\Lambda f = \sum_{i\in I} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}(f), \qquad \forall f \in H.$$

Is called $g-$fusion frame operator. It can be easily verify that

$$\langle S_\Lambda f, f\rangle = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}(f), \Lambda_i P_{W_i}(f)\rangle, \qquad \forall f \in H. \tag{0.2}$$

Furthermore, if Λ is a $g-$fusion frame with bounds A and B, then

$$A\langle f, f\rangle \leq \langle S_\Lambda f, f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H.$$

It easy to see that the operator S_Λ is bounded, self-adjoint, positive, now we proof the inversibility of S_Λ. Let $x \in H$ we have

$$||U_\Lambda(f)|| = ||\{v_i \Lambda_i P_{W_i}(f)\}_{i\in I}|| = ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}(f), \Lambda_i P_{W_i}(f)\rangle||^{\frac{1}{2}}.$$

Since Λ is $g-$fusion frame then

$$\sqrt{A}||\langle f, f\rangle||^{\frac{1}{2}} \leq ||U_\Lambda f||.$$

Then

$$\sqrt{A}||f|| \leq ||U_\Lambda f||.$$

Frome lemma 0.2, T_Λ is surjective and by lemma 0.3, $T_\Lambda U_\Lambda = S_\Lambda$ is invertible. We now, $AI_H \leq S_\Lambda \leq BI_H$ and this gives $B^{-1}I_H \leq S_\Lambda^{-1} \leq A^{-1}I_H$.

Definition 0.7. [10] A family $\left\{\{f_{i,j}\}_{i\in\mathbb{I}}\right\}_{j\in[m]}$ of frames for H is called woven if there exist universal constants $0 < A < B < \infty$ such that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$, the family $\{f_{i,j}\}_{i\in\sigma_j, j\in[m]}$ is a frame for H with lower and upper frame bounds A and B, respectively. Each family $\{f_{i,j}\}_{i\in\sigma_j, j\in[m]}$ is called a weaving. Where $[m] = \{1, ..., m\}$

1. Woven $g-$fusion frame in Hilbert C^*-module

Now, we define the notion of woven $g-$fusion frame in Hilbert C^*-module.

Definition 1.1. A family of $g-$fusion frames $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{i\in\mathbb{I}}$ for $j \in [m]$, is said woven $g-$fusion frames if there exist universal constants A and B, such that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$, the family $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a $g-$fusion frame for H with lower and upper frame bounds A and B. Each family $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{i\in\sigma_j, j\in[m]}$ is called a weaving $g-$fusion frame.

For any partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$, we define the operator

$$S_\Lambda^{\sigma_j} f = \sum_{i\in\sigma_j} v_i^2 P_{W_i}\Lambda_i\Lambda_i^* P_{W_i} f, \quad \forall f \in H.$$

2. Main Results

The following Theorem characterize woven $g-$frames. That we will used in the proof of the next results.

Theorem 1.2. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}$ be two $g-$fusion frame for H, then for every partition σ of $\mathbb{I}$, Λ and Γ are woven $g-$fusion frame for H if and only if*

$$A\|f\|^2 \leq \left\|\sum_{i\in\sigma} v_i^2 \langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c} \mu_i^2 \langle\Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle\right\| \leq B\|f\|^2, \tag{1.1}$$

for some $A, B > 0$.

Proof. Suppose that Λ and Γ are woven $g-$fusion frame for H with $g-$fusion frame bounds A and B, then for each $f \in H$

$$A\|f\|^2 \leq \left\|\sum_{i\in\sigma} v_i^2 \langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c} \mu_i^2 \langle\Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle\right\| \leq B\|f\|^2,$$

For the converse, we have for each $f \in H$

$$\begin{aligned}\left\|\sum_{i\in\sigma} v_i^2\langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\mu_i^2\langle\Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle\right\| &= \|\langle S_\Lambda^\sigma f, f\rangle + \langle S_\Gamma^{\sigma^c} f, f\rangle\| \\ &= \|\langle(S_\Lambda^\sigma + S_\Gamma^{\sigma^c})f, f\rangle\| \\ &= \|\langle(S_\Lambda^\sigma + S_\Gamma^{\sigma^c})^{\frac{1}{2}} f, (S_\Lambda^\sigma + S_\Gamma^{\sigma^c})^{\frac{1}{2}} f\| \\ &= \|(S_\Lambda^\sigma + S_\Gamma^{\sigma^c})^{\frac{1}{2}} f\|^2,\end{aligned}$$

since,

$$A\|f\|^2 \leq \|(S_\Lambda^\sigma + S_\Gamma^{\sigma^c})^{\frac{1}{2}} f\|^2 \leq B\|f\|^2,$$

by lemma 0.4, there exists $\lambda, \mu > 0$ such that

$$\lambda\langle f, f\rangle \leq \langle(S_\Lambda^\sigma + S_\Gamma^{\sigma^c})f, f\rangle \leq \mu\langle f, f\rangle,$$

then,

$$\lambda\langle f, f\rangle \leq \sum_{i\in\sigma} v_i^2\langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\mu_i^2\langle\Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle \leq \mu\langle f, f\rangle.$$

So, Λ and Γ are woven $g-$fusion frame for H. □

In the next we constructed some new woven $g-$frames in Hilbert C^*-modules.

Theorem 1.3. *Suppose $\{\Lambda_i \in End_{\mathcal{A}}^*(H, H_i)\}_{i\in\mathbb{I}}$, $\{\Gamma_i \in End_{\mathcal{A}}^*(H, H_i)\}_{i\in\mathbb{I}}$ and for every $i \in \mathbb{I}$, $\mathbb{J}_i$ is subset of index set $\mathbb{I}$ and $\{v_i\}_{i\in\mathbb{I}}$, $\{\mu_i\}_{i\in\mathbb{I}}$ are family of weights in $\mathcal{A}$. Let $\{f_{i,j}\}_{j\in\mathbb{J}_i}$ and $\{g_{i,j}\}_{j\in\mathbb{J}_i}$ be frame sequences in H_i with frame bounds (A_{f_i}, B_{f_i}) and (A_{g_i}, B_{g_i}), respectively. Define*

$$W_i = \overline{span}\{\Lambda_i^* f_{i,j}\}_{j\in\mathbb{J}_i}, \quad V_i = \overline{span}\{\Gamma_i^* g_{i,j}\}_{j\in\mathbb{J}_i}.$$

Suppose that

$$0 < A_f = \inf_{i\in\mathbb{I}} A_{f_i} \leq B_f = \sup_{i\in\mathbb{I}} B_{f_i} < \infty,$$

and

$$0 < A_g = \inf_{i\in\mathbb{I}} A_{g_i} \leq B_g = \sup_{i\in\mathbb{I}} B_{g_i} < \infty.$$

Then the following conditions are equivalent:

(1) *$\{v_i\Lambda_i^* f_{i,j}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ and $\{\mu_i\Gamma_i^* g_{i,j}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ are woven frames in H.*

(2) *$\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are woven $g-$fusion frames in H.*

Proof. Since for every $i \in \mathbb{I}$, $\{f_{i,j}\}_{j\in\mathbb{J}_i}$ and $\{g_{i,j}\}_{j\in\mathbb{J}_i}$ be frame sequences in H_i with frame bounds (A_{f_i}, B_{f_i}) and (A_{g_i}, B_{g_i}), respectively. Then for $\sigma \subset \mathbb{I}$,

$$\begin{aligned}
&A_f \sum_{i\in\sigma} v_i^2 \langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + A_g \sum_{i\in\sigma^c} \mu_i^2 \langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle \\
&\leq \sum_{i\in\sigma} A_{f_i} v_i^2 \langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c} A_{g_i} \mu_i^2 \langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle \\
&= \sum_{i\in\sigma} A_{f_i} \langle v_i\Lambda_i P_{W_i} f, v_i\Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c} A_{g_i} \langle \mu_i\Gamma_i P_{V_i} f, \mu_i\Gamma_i P_{V_i} f\rangle \\
&\leq \sum_{i\in\sigma}\sum_{i\in\mathbb{J}_i} \langle v_i\Lambda_i P_{W_i} f, f_{i,j}\rangle\langle f_{i,j}, v_i\Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\sum_{i\in\mathbb{J}_i} \langle \mu_i\Gamma_i P_{V_i} f, g_{i,j}\rangle\langle g_{i,j}, \mu_i\Gamma_i P_{V_i} f\rangle \\
&\leq \sum_{i\in\sigma} B_{f_i} \langle v_i\Lambda_i P_{W_i} f, v_i\Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c} B_{g_i} \langle \mu_i\Gamma_i P_{V_i} f, \mu_i\Gamma_i P_{V_i} f\rangle \\
&\leq B_f \sum_{i\in\sigma} \langle v_i\Lambda_i P_{W_i} f, v_i\Lambda_i P_{W_i} f\rangle + B_g \sum_{i\in\sigma^c} \langle \mu_i\Gamma_i P_{V_i} f, \mu_i\Gamma_i P_{V_i} f\rangle.
\end{aligned}$$

(1) $\Longrightarrow$ (2) Let $\{v_i\Lambda_i^* f_{i,j}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ and $\{\mu_i\Gamma_i^* g_{i,j}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ be woven frames for H with universal frame bounds C and D, the above calculation shows that for every $f \in H$,

$$\begin{aligned}
&\sum_{i\in\sigma} v_i^2 \langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c} \mu_i^2 \langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle \\
&\leq \frac{1}{A}\Bigg(\sum_{i\in\sigma}\sum_{i\in\mathbb{J}_i} \langle v_i\Lambda_i P_{W_i} f, f_{i,j}\rangle\langle f_{i,j}, v_i\Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\sum_{i\in\mathbb{J}_i} \langle \mu_i\Gamma_i P_{V_i} f, g_{i,j}\rangle\langle g_{i,j}, \mu_i\Gamma_i P_{V_i} f\rangle \Bigg) \\
&= \frac{1}{A}\Bigg(\sum_{i\in\sigma}\sum_{i\in\mathbb{J}_i} \langle f, v_i\Lambda_i^* f_{i,j}\rangle\langle v_i\Lambda_i^* f_{i,j}, f\rangle + \sum_{i\in\sigma^c}\sum_{i\in\mathbb{J}_i} \langle f, \mu_i\Gamma_i^* g_{i,j}\rangle\langle \mu_i\Gamma_i^* g_{i,j}, f\rangle \Bigg) \\
&\leq \frac{D}{A}\langle f, f\rangle,
\end{aligned}$$

where $A = \min\{A_f, A_g\}$. For lower frame bound,

$$\begin{aligned}
&\sum_{i\in\sigma} v_i^2 \langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c} \mu_i^2 \langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle \\
&\geq \frac{1}{B}\Bigg(\sum_{i\in\sigma}\sum_{i\in\mathbb{J}_i} \langle v_i\Lambda_i P_{W_i} f, f_{i,j}\rangle\langle f_{i,j}, v_i\Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\sum_{i\in\mathbb{J}_i} \langle \mu_i\Gamma_i P_{V_i} f, g_{i,j}\rangle\langle, g_{i,j}\mu_i\Gamma_i P_{V_i} f\rangle \Bigg) \\
&= \frac{1}{B}\Bigg(\sum_{i\in\sigma}\sum_{i\in\mathbb{J}_i} \langle f, v_i\Lambda_i^* f_{i,j}\rangle\langle v_i\Lambda_i^* f_{i,j}, f\rangle + \sum_{i\in\sigma^c}\sum_{i\in\mathbb{J}_i} \langle f, \mu_i\Gamma_i^* g_{i,j}\rangle\langle \mu_i\Gamma_i^* g_{i,j}, f\rangle \Bigg) \\
&\geq \frac{C}{B}\langle f, f\rangle
\end{aligned}$$

where $B = \max\{B_f, B_g\}$

(2) $\implies$ (1) Let $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ be woven $g-$fusion frames with universel frame bounds C and D. Then for every $f \in H$, we have

$$\begin{aligned}
&\sum_{i\in\sigma}\sum_{i\in\mathbb{J}_i}\langle f, v_i\Lambda_i^* f_{i,j}\rangle\langle v_i\Lambda_i^* f_{i,j}, f\rangle + \sum_{i\in\sigma^c}\sum_{i\in\mathbb{J}_i}\langle f, \mu_i\Gamma_i^* g_{i,j}\rangle\langle \mu_i\Gamma_i^* g_{i,j}, f\rangle\\
&= \sum_{i\in\sigma}\sum_{i\in\mathbb{J}_i}\langle v_i\Lambda_i P_{W_i} f, f_{i,j}\rangle\langle f_{i,j}, v_i\Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\sum_{i\in\mathbb{J}_i}\langle \mu_i\Gamma_i P_{V_i} f, g_{i,j}\rangle\langle, g_{i,j}\mu_i\Gamma_i P_{V_i} f\rangle\\
&\geq \sum_{i\in\sigma} A_{f_i} v_i^2\langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c} A_{g_i}\mu_i^2\langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle\\
&\geq A\Big(\sum_{i\in\sigma} v_i^2\langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\mu_i^2\langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle\Big)\\
&\geq AC\langle f, f\rangle.
\end{aligned}$$

And similary

$$\sum_{i\in\sigma}\sum_{i\in\mathbb{J}_i}\langle f, v_i\Lambda_i^* f_{i,j}\rangle\langle v_i\Lambda_i^* f_{i,j}, f\rangle + \sum_{i\in\sigma^c}\sum_{i\in\mathbb{J}_i}\langle f, \mu_i\Gamma_i^* g_{i,j}\rangle\langle \mu_i\Gamma_i^* g_{i,j}, f\rangle \leq BD\langle f, f\rangle.$$

□

Theorem 1.4. *Let K be a closed orthogonaly complemented subspace of H and let $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ be woven $g-$fusion frame for H with woven bounds A and B. Then $\{W_i \cap K, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i \cap K, \Gamma_i, \mu_i\}$ are woven $g-$fusion frames for K with universal bounds A and B.*

Proof. Let the operators $P_{W_i\cap K} = P_{W_i}(P_K)$ and $P_{V_i\cap K} = P_{V_i}(P_K)$ be orthogonal projections of H onto $W_i \cap K$ and $V_i \cap K$, respectively. Then for every $f \in K$, we can write:

$$\begin{aligned}
&\sum_{i\in\sigma} v_i^2\langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\mu_i^2\langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle\\
&= \sum_{i\in\sigma} v_i^2\langle \Lambda_i P_{W_i} P_K f, \Lambda_i P_{W_i} P_K f\rangle + \sum_{i\in\sigma^c}\mu_i^2\langle \Gamma_i P_{V_i} P_K f, \Gamma_i P_{V_i} P_K f\rangle\\
&= \sum_{i\in\sigma} v_i^2\langle \Lambda_i P_{W_i\cap K} f, \Lambda_i P_{W_i\cap K} f\rangle + \sum_{i\in\sigma^c}\mu_i^2\langle \Gamma_i P_{V_i\cap K} f, \Gamma_i P_{V_i\cap K} f\rangle.
\end{aligned}$$

we conclude the result. □

Theorem 1.5. *Let $\{W_{i,j}, \Lambda_{i,j}, v_{i,j}\}_{i\in\mathbb{I}}$ be a $g-$fusion bessel sequence of subspaces for H with bounds B_j for all $j \in [m]$. Then every weaving of this sequence is a bessel sequence.*

Proof. For every partition $\{\sigma_j\}_{j\in[m]}$, such that $\sigma_j \in \mathbb{I}$ for $j \in [m]$ and for $f \in H$, we have

$$\sum_{j=1}^{m}\sum_{i\in\sigma_j} v_i^2\langle\Lambda_{i,j}P_{W_{i,j}}f, \Lambda_{i,j}P_{W_{i,j}}f\rangle \leq \sum_{j=1}^{m}\sum_{i=1}^{\infty} v_i^2\langle\Lambda_{i,j}P_{W_{i,j}}f, \Lambda_{i,j}P_{W_{i,j}}f\rangle \leq \sum_{j=1}^{m} B_j\langle f, f\rangle.$$

□

Theorem 1.6. *Suppose that $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are $g-$fusion frames for H and also let for every two disjoint finite sets $I, J \subseteq \mathbb{I}$ and every $\epsilon > 0$, there exist subsets $\sigma, \delta \subseteq \mathbb{I}(I \cup J)$ such that the lower $g-$fusion frame bound of $\{W_i, \Lambda_i, v_i\}_{i\in(I\cup\sigma)} \cup \{V_i, \Gamma_i, \mu_i\}_{i\in(J\cup\delta)}$ is less than ϵ. Then there exists $\mathcal{M} \subseteq I$ such that $\{W_i, \Lambda_i, v_i\}_{i\in\mathcal{M}} \cup \{V_i, \Gamma_i, \mu_i\}_{i\in\mathcal{M}^c}$ is not a $g-$fusion frame. Hence $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are not woven $g-$fusion frames.*

Proof. Let $\epsilon > 0$ be arbitrary. By hypothesis, for $I_0 = J_0 = \emptyset$, we can choose $\sigma_1 \subset \mathbb{I}$, so that if $\delta_1 = \sigma_1^c$, then the lower $g-$fusion frame bound of $\{W_i, \Lambda_i, v_i\}_{i\in(I\cup\sigma_1)} \cup \{V_i, \Gamma_i, \mu_i\}_{i\in(J\cup\delta_1)}$ is less than ϵ. Thus there exists $f_1 \in H$, with $\langle f_1, f_1\rangle = 1$ such that

$$\sum_{i\in\sigma_1} v_i^2\langle\Lambda_iP_{W_i}f_1, \Lambda_iP_{W_i}f_1\rangle + \sum_{i\in\delta_1} \mu_i^2\langle\Gamma_iP_{V_i}f_1, \Gamma_iP_{V_i}f_1\rangle < \epsilon.$$

Since $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are $g-$fusion frames for H, so

$$\sum_{i=1}^{\infty} v_i^2\langle\Lambda_iP_{W_i}f_1, \Lambda_iP_{W_i}f_1\rangle + \sum_{i=1}^{\infty} \mu_i^2\langle\Gamma_iP_{V_i}f_1, \Gamma_iP_{V_i}f_1\rangle < \infty$$

therefor there is a positive integer k_1 such that

$$\sum_{i=k_1+1}^{\infty} v_i^2\langle\Lambda_iP_{W_i}f_1, \Lambda_iP_{W_i}f_1\rangle + \sum_{i=k_1+1}^{\infty} \mu_i^2\langle\Gamma_iP_{V_i}f_1, \Gamma_iP_{V_i}f_1\rangle < \infty$$

Let $I_1 = \sigma_1 \cap [k_1]$ and $J_1 = \delta_1 \cap [k_1]$. Then $I_1 \cap J_1 = \emptyset$ and $I_1 \cup J_1 = [k_1]$. By assumption, there are subsets $\sigma_2, \delta_2 \subset [k_1]^c$ with $\delta_2 = [k_1]^c - \sigma_2$ such that the lower fusion frame bound of $\{W_i, \Lambda_i, v_i\}_{i\in(I\cup\sigma_2)} \cup \{V_i, \Gamma_i, \mu_i\}_{i\in(J\cup\delta_2)}$ is less than $\frac{\epsilon}{2}$, so there exists a vector $f_2 \in H$ with $\langle f_2, f_2\rangle = 1$, such that

$$\sum_{i\in I_1\cup\sigma_2} v_i^2\langle\Lambda_iP_{W_i}f_2, \Lambda_iP_{W_i}f_2\rangle + \sum_{i\in J_1\cup\delta_2} \mu_i^2\langle\Gamma_iP_{V_i}f_2, \Gamma_iP_{V_i}f_2\rangle < \frac{\epsilon}{2}.$$

Similarly, there is $k_2 > k_1$ such that

$$\sum_{i=k_2+1}^{\infty} v_i^2 \langle \Lambda_i P_{W_i} f_2, \Lambda_i P_{W_i} f_2 \rangle + \sum_{i=k_2+1}^{\infty} \mu_i^2 \langle \Gamma_i P_{V_i} f_2, \Gamma_i P_{V_i} f_2 \rangle < \frac{\epsilon}{2}.$$

Set $I_2 = I_1 \cup (\sigma_2 \cap [k_2])$ and $J_2 = J_1 \cup (\delta_2 \cap [k_2])$. Note that $I_2 \cap J_2 = \emptyset$ and $I_2 \cup J_2 = [k_2]$. Thus by induction, there are

(1) a sequence of natural numbers $k_{i i \in \mathbb{I}}$ with $k_i < k_{i+1}$ for all $i \in \mathbb{I}$,

(2) a sequence of vectors $\{f_i\}_{i \in I}$ from H with $\langle f_i, f_i \rangle = 1$ for all $i \in \mathbb{I}$,

(3) subsets $\sigma_i \subset [k_{i-1}]^c$, $\delta_i = [k_{i-1}]^c - \sigma_i, i \in \mathbb{I}$ and

(4) $I_i = I_{i-1} \cup (\sigma_i \cap [k_i]),\ J_i = J_{i-1} \cup (\delta_i \cap [k_i]), i \in \mathbb{I}$ wich are abiding both:

$$\sum_{i \in I_{n-1} \cup \sigma_n} v_i^2 \langle \Lambda_i P_{W_i} f_n, \Lambda_i P_{W_i} f_n \rangle + \sum_{i \in J_{n-1} \cup \delta_n} \mu_i^2 \langle \Gamma_i P_{V_i} f_n, \Gamma_i P_{V_i} f_n \rangle < \frac{\epsilon}{n},$$

and

$$\sum_{i=k_n+1}^{\infty} v_i^2 \langle \Lambda_i P_{W_i} f_n, \Lambda_i P_{W_i} f_n \rangle + \sum_{i=k_n+1}^{\infty} \mu_i^2 \langle \Gamma_i P_{V_i} f_n, \Gamma_i P_{V_i} f_n \rangle < \frac{\epsilon}{n}$$

By construction $I_i \cup J_i = \emptyset$ and $I_i \cup J_i = [k_i]$, if we suppose that $\mathcal{M} = \cup_{i=1}^{\infty} I_i$ then $\mathcal{M}^c = \cup_{i=1}^{\infty} J_i$ such that $\mathcal{M} \cup \mathcal{M}^c = \mathbb{I}$, then we conclude from the above inequalities:

$$\begin{aligned}
&\sum_{i \in \mathcal{M}} v_i^2 \langle \Lambda_i P_{W_i} f_i, \Lambda_i P_{W_i} f_i \rangle + \sum_{i \in \mathcal{M}^c} \mu_i^2 \langle \Gamma_i P_{V_i} f_i, \Gamma_i P_{V_i} f_i \rangle \\
&= \left(\sum_{i \in I_n} \langle \Lambda_i P_{W_i} f_i, \Lambda_i P_{W_i} f_i + \sum_{i \in J_n} \langle \mu_i^2 \Gamma_i P_{V_i} f_i, \Gamma_i P_{V_i} f_i \rangle \right) \\
&+ \left(\sum_{i \in \mathcal{M} \cap [k_n]^c} v_i^2 \langle \Lambda_i P_{W_i} f_i, \Lambda_i P_{W_i} f_i \rangle + \sum_{i \in \mathcal{M}^c \cap [k_n]^c} \mu_i^2 \langle \Gamma_i P_{V_i} f_i, \Gamma_i P_{V_i} f_i \rangle \right) \\
&\leq \left(\sum_{i \in I_{n-1} \cup \sigma_n} v_i^2 \langle \Lambda_i P_{W_i} f_n, \Lambda_i P_{W_i} f_n \rangle + \sum_{i \in J_{n-1} \cup \delta_n} \mu_i^2 \langle \Gamma_i P_{V_i} f_n, \Gamma_i P_{V_i} f_n \rangle \right) \\
&+ \left(\sum_{i=k_n+1}^{\infty} v_i^2 \langle \Lambda_i P_{W_i} f_i, \Lambda_i P_{W_i} f_i \rangle + \sum_{i=k_n+1}^{\infty} \mu_i^2 \langle \Gamma_i P_{V_i} f_i, \Gamma_i P_{V_i} f_i \rangle \right) \\
&< \frac{\epsilon}{n} + \frac{\epsilon}{n} = \frac{2\epsilon}{n}.
\end{aligned}$$

Therfore the lower $g-$fusion frame of $\{W_i, \Lambda_i, v_i\}_{i \in \mathcal{M}} \cup \{V_i, \Gamma_i, \mu_i\}_{i \in \mathcal{M}^c}$ is zero, that is a contradiction. thus $\{W_i, \Lambda_i, v_i\}_{i \in \mathbb{I}} \cup \{V_i, \Gamma_i, \mu_i\}_{i \in \mathbb{I}}$ can not be a woven $g-$fusion frame. □

Theorem 1.7. *Suppose that $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are $g-$fusion frames for H with optimal upper $g-$fusion frame bounds B_1 and B_2 such that they be woven $g-$fusion frames. Then $B_1 + B_2$ can not be the optimal upper woven bound.*

Proof. Assume on the contrary, which is $B_1 + B_2$ is the smallest upper weaving bound for all possible weavings. Then by definition of optimal upper bound, we can choose $\sigma \subset \mathbb{I}$ and $\langle f, f\rangle = 1$, such that

$$\sup_{\langle f,f\rangle=1}\left(\sum_{i\in\sigma} v_i^2\langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\mu_i^2\langle\Lambda_i P_{V_i} f, \Lambda_i P_{V_i} f\rangle\right) = B_1 + B_2.$$

Using of supreme property, for every $\epsilon > 0$, there exists $f \in H$, such that

$$\sum_{i\in\sigma} v_i^2\langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\mathbb{I}}\mu_i^2\langle\Lambda_i P_{V_i} f, \Lambda_i P_{V_i} f\rangle \geq B_1 + B_2 - \epsilon,$$

and using of upper fusion frame property, we have

$$\sum_{i\in\sigma} v_i^2\langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\mathbb{I}}\mu_i^2\langle\Lambda_i P_{V_i} f, \Lambda_i P_{V_i} f\rangle \leq B_1 + B_2.$$

So,

$$\sum_{i\in\mathbb{I}-\sigma} v_i^2\langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\mathbb{I}-\sigma^c}\mu_i^2\langle\Lambda_i P_{V_i} f, \Lambda_i P_{V_i} f\rangle \leq \epsilon.$$

Now, if we assume that is a weaving for which the lower frame bound approaches zero. Theorem 1.6 gives that $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are not woven $g-$fusion frame, which is a contradiction. □

Theorem 1.8. *Let $\{W_i, \Lambda_i, v_i\}_{i\in J}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in J}$ be $g-$fusion frames, such that $J \subset \mathbb{I}$. Then $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are woven $g-$fusion frames.*

Proof. Let the positive constants A be the lower woven bound for $\{W_i, \Lambda_i, v_i\}_{i\in J}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in J}$. Then for every $\sigma \subset \mathbb{I}$ and $f \in H$, we have

$$\begin{aligned}
A\langle f, f\rangle &\leq \sum_{i\in\sigma\cap J} v_i^2\langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c\cap J}\mu_i^2\langle\Lambda_i P_{V_i} f, \Lambda_i P_{V_i} f\rangle\\
&\leq \sum_{i\in\sigma} v_i^2\langle\Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma^c}\mu_i^2\langle\Lambda_i P_{V_i} f, \Lambda_i P_{V_i} f\rangle\\
&\leq (B_\Lambda + B_\Gamma)\langle f, f\rangle,
\end{aligned}$$

where B_Λ and B_Γ are upper fusion frame bounds for $\{W_i, \Lambda_i, v_i\}_{i\in J}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in J}$ respectively. □

2. Perturbation of woven $g-$fusion frames

The question of stability plays an important role in various fields of applied mathematics. The classical theorem of the stability of a base is due to Paley and Wiener [16]. It is based on the fact that a bounded operator T on a Banach space is invertible if $\|I-T\|<1$.

The following theorem is a Paley–Wiener type stability theorem for woven $g-$frames in Hilbert C^*-modules.

Theorem 2.1. *Let $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ be $g-$fusion frames for H with $g-$fusion frame bounds (A_Λ, B_Λ) and (A_Γ, B_Γ), respectively. If there exist constants $0<\lambda_1, \lambda_2, \mu<1$ such that:*

$$\frac{2}{A_\Lambda}\left(\sqrt{B_\Lambda}+\sqrt{B_\Gamma}\right)\left(\lambda_1\sqrt{B_\Lambda}+\lambda_2\sqrt{B_\Gamma}+\mu\right)\leq 1$$

and

$$\|T_\Lambda f - T_\Gamma f\| \leq \lambda_1\|T_\Lambda f\| + \lambda_2\|T_\Gamma f\| + \mu,$$

where T_Λ, T_Γ are the synthesis operators for these $g-$fusion frames, then $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are woven $g-$fusion frames.

Proof. for each $\sigma\subset\mathbb{I}$, we define the bounded operators

$$T_\Lambda^\sigma : l^2(\{H_i\}_{i\in\sigma}) \to H, \quad T_\Lambda^\sigma(f) = \sum_{i\in\sigma} v_i P_{W_i}\Lambda_i^* f_i,$$

and

$$T_\Gamma^\sigma : l^2(\{H_i\}_{i\in\sigma}) \to H, \quad T_\Gamma^\sigma(f) = \sum_{i\in\sigma} \mu_i P_{V_i}\Gamma_i^* f_i,$$

for every $f=\{f_i\}_{i\in I}\in l^2(\{H_i\}_{i\in\sigma})$. Note that

$$\|T_\Lambda^\sigma(f)\| \leq \|T_\Lambda(f)\|, \quad \|T_\Gamma^\sigma(f)\| \leq \|T_\Gamma(f)\|,$$

and

$$\|T_\Lambda^\sigma(f) - T_\Gamma^\sigma(f)\| \leq \|T_\Lambda(f) - T_\Gamma(f)\|.$$

For every $f \in H$ and $\sigma \subset \mathbb{I}$, we have

$$\begin{aligned}\|T_\Lambda^\sigma U_\Lambda^\sigma f - T_\Gamma^\sigma U_\Gamma^\sigma f\| &= \|T_\Lambda^\sigma U_\Lambda^\sigma f - T_\Lambda^\sigma U_\Gamma^\sigma f + T_\Lambda^\sigma U_\Gamma^\sigma f - T_\Gamma^\sigma U_\Gamma^\sigma f\| \\ &\leq \|T_\Lambda^\sigma (U_\Lambda^\sigma - U_\Gamma^\sigma) f\| + \|(T_\Lambda^\sigma - T_\Gamma^\sigma) U_\Gamma^\sigma f\| \\ &\leq \|T_\Lambda\|\|T_\Lambda - T_\Gamma\|\|f\| + \|T_\Lambda - T_\Gamma\|\|T_\Gamma\|\|f\| \\ &= \|T_\Lambda - T_\Gamma\|\Big(\|T_\Lambda\| - \|T_\Gamma\|\Big)\|f\| \\ &\leq \Big(\lambda_1\sqrt{B_\Lambda} + \lambda_2\sqrt{B_\Gamma} + \mu\Big)\Big(\sqrt{B_\Lambda} + \sqrt{B_\Gamma}\Big)\|f\| \\ &\leq \frac{A_\Lambda}{2}\|f\|.\end{aligned}$$

Now by using above calculation, we have

$$\begin{aligned}S_\Lambda^{\sigma^c} + S_\Gamma^\sigma &= S_\Lambda + S_\Gamma^\sigma - S_\Lambda^\sigma \\ &\geq A_\Lambda I - \|S_\Lambda^\sigma - S_\Gamma^\sigma\| I \\ &\geq A_\Lambda I - \frac{A_\Lambda}{2} I \\ &= \frac{A_\Lambda}{2} I.\end{aligned}$$

This shows that $\frac{A_\Lambda}{2}$ is the universal lower woven bound. Finally, for universal upper bound, we have

$$\begin{aligned}\sum_{i\in\sigma^c} v_i^2\langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle + \sum_{i\in\sigma}\mu_i^2\langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle &\leq \sum_{i\in\mathbb{I}} v_i^2\langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f\rangle \\ &+ \sum_{i\in\mathbb{I}}\mu_i^2\langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f\rangle \\ &\leq (B_\Lambda + B_\Gamma)\|f\|.\end{aligned}$$

□

Theorem 2.2. *Let $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ be $g-$fusion frames for H with $g-$fusion frame bounds (A_Λ, B_Λ) and (A_Γ, B_Γ), respectively. If there exist constants $0 < \lambda, \mu, \gamma < 1$, such that $\lambda B_\Lambda + \mu B_\Gamma + \gamma\sqrt{B_\Lambda} < A_\Lambda$. We have*

$$S_\Lambda^\sigma < \lambda S_\Lambda^\sigma + \mu S_\Gamma^\sigma + \gamma U_\Lambda^\sigma,$$

where S_Λ, U_Λ are $g-$fusion frame operators of $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$. Then $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are woven $g-$fusion frame with universal woven bounds

$$\left(A_\Lambda - \lambda B_\Lambda - \mu B_\Gamma - \gamma\sqrt{B_\Lambda}\right), \quad \left(B_\Gamma + \lambda B_\Lambda + \mu B_\Gamma + \gamma\sqrt{B_\Lambda}\right).$$

Proof. First, for lower frame bound, we have

$$\begin{aligned} S_\Lambda^\sigma + S_\Gamma^{\sigma^c} &= S_\Lambda + S_\Gamma^{\sigma^c} - S_\Lambda^{\sigma^c} \\ &= S_\Lambda - \left(S_\Lambda^{\sigma^c} - S_\Gamma^{\sigma^c}\right) \\ &\geq A_\Lambda I - \left(\lambda S_\Lambda^{\sigma^c} + \mu S_\Gamma^{\sigma^c} + \gamma U_\Lambda^{\sigma^c}\right) \\ &\geq \left(A_\Lambda - \lambda B_\Lambda - \mu B_\Gamma - \gamma\sqrt{B_\Lambda}\right) I. \end{aligned}$$

Also, for upper frame bound, we have

$$\begin{aligned} S_\Lambda^\sigma + S_\Gamma^{\sigma^c} &= S_\Gamma + S_\Lambda^\sigma - S_\Gamma^\sigma \\ &\leq \left(B_\Gamma + \lambda B_\Lambda + \mu B_\Gamma + \gamma\sqrt{B_\Lambda}\right) I \end{aligned}$$

Therefore $g-$fusion frames $\{W_i, \Lambda_i, v_i\}_{i\in\mathbb{I}}$ and $\{V_i, \Gamma_i, \mu_i\}_{i\in\mathbb{I}}$ are woven $g-$fusion frame with considered bounds. □

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