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Generalized rational α_*-contraction in C^*-algebra valued b-metric spaces

Mohamed Rossafi[1], **Hafida Massit**[2], **Abdelkarim Kari**[3]

[1]Laboratory of Partial Differential Equations, Spectral Algebra and Geometry, Department of Mathematics, Faculty of Sciences, University of Ibn Tofail, P. O. Box 133 Kenitra, Morocco

[2]LaSMA Laboratory Department of Mathematics Faculty of Sciences, Dhar El Mahraz University Sidi Mohamed Ben Abdellah, P. O. Box 1796 Fez Atlas, Morocco

[3]Laboratory of Analysis, Modeling and Simulation, Faculty of Sciences Ben M'Sik, Hassan II University, Casablanca, Morocco

E-mail: [1]rossafimohamed@gmail.com, [2]massithafida@yahoo.fr, [3]abdkrimkariprofes@gmail.com

Abstract

This present paper extends some common fixed point theorems for generalized rational α_*-contraction of multi-valued mappings in the setting of C^*-algebra valued $b-$metric spaces.

Keywords: Fixed point, generalized rational α_*-contraction, multi-valued mapping, Picard sequences, C^*-algebra valued b-metric spaces.

1. Introduction

The concept of multi-valued contraction mappings was introduced by Nadler[7], he etablished that a multi-valued contraction mapping has a fixed point in a complete metric spaces.

Recently, Ma et al. [4] announced the notion of C^*-algebra valued metric space and formulated some first fixed point theorems in the C^*-algebra valued metric space.

Many authors initiated and studied many existing fixed point theorems in such spaces, see [5, 6, 8].

Very recently, Amer [1] in 2017 introduced a new concept known as generalized $\alpha_* - \psi-$Geraghty contraction type for multivalued mappings.

In this paper, we provide some fixed point results for generalized rational α_*-contraction for multi-valued mappings in C^*-algebra valued $b-$metric spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{A}$ an unital (i.e. have an unity element I) C^*-algebra with linear involution $*$, such that for all $x, y \in \mathbb{A}$,

$$(xy)^* = y^*x^*,\text{and } x^{**} = x.$$

We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$

if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$,where $\sigma(x)$ is the spectrum of x.Using positive element ,we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows :

$$x \preceq y \text{ if and only if } y - x \succeq \theta$$

where θ means the zero element in $\mathbb{A}$.

we denote the set $x \in \mathbb{A} : x \succeq \theta$ by $\mathbb{A}_+$ and $\|x\| = (x^*x)^{\frac{1}{2}}$.

and $\mathbb{A}'$ will denote the set $\{a \in \mathbb{A}_+; ab = ba, \forall b \in \mathbb{A}\}$

Now, we recollect some definitions and lemmas which will be useful in our main results.

Lemma 0.1. [6] *Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I,*

(1) *for any $x \in \mathbb{A}_+$ we have $x \preceq I \iff \|x\| \leq 1$,*
(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$ then $I - a$ is invertible and $\|a(1-a)^{-1}\| < 1$,*
(3) *Suppose that $a, b \in \mathbb{A}_+$ and $ab = ba$, then $ab \succeq \theta$,*
(4) *Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$, with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'_+$ is invertible operator, then $(I-a)^{-1}b \succeq (I-a)^{-1}c$.*

Definition 0.2. [8] Let X be a non-empty set, $b \in \mathbb{A}$ and $b \succeq I$.

Suppose the mapping $d : X \times X \to \mathbb{A}_+$ satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
(iii) $d(x, y) \preceq b[d(x, u) + d(u, y)]$ for all $x, y, u \in X$.

Then $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued $b-$metric space with coefficient b.

Example 0.3. Let $X = [-1, 1]$ and $A = \mathbb{M}_2(\mathbb{R})$. Define partial ordering on $\mathbb{A}$ as

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$\Leftrightarrow a_i \succeq b_i \ for \ i = 1, 2, 3, 4.$

Define $d : X \times X \to \mathbb{M}_2(\mathbb{R})+$ by

$$d(x, y) = \begin{pmatrix} |x-y|^2 & 0 \\ 0 & |x-y|^2 \end{pmatrix}$$

It is easy to verify d is a C^*-algebra-valued $b-$ metric with a coefficient $b = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $(X, \mathbb{A}, d)$ is a complete C^*- algebra-valued $b-$metric space.

Lemma 0.4. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$ metric space with $b \succeq I$.*

Suppose that $\{x_n\}$ a sequence in X, such that

$$d(x_{n+1}, x_n) \preceq \delta d(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$ and $\delta \in [0, 1)$.

Then $\{x_n\}$ is a Cauchy sequence.

Proof. First let us note that

$$d(x_{n+1}, x_n) \preceq \delta^n d(x_1, x_0) \ \forall n \in \mathbb{N}$$

we have for $m \geq 1$, $p \geq 1$

$$\begin{aligned} d(x_m, x_{m+p}) &\preceq b\left(d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})\right) \\ &\preceq bd(x_m, x_{m+1}) + b^2 d(x_{m+1}, x_{m+2}) + ... + b^{p-1}(d(x_{m+p-2}, x_{m+p-1}) + b^{p-1}d(x_{m+p-1}, x_{m+p}) \\ &\preceq b\delta^m d(x_0, x_1) + b\delta^{m+1} d(x_0, x_1) + b^2\delta^{m+2} d(x_0, x_1) + b^2\delta^{m+3} d(x_0, x_1) \\ &+ ...... + b^{p-1}\delta^{m+p} d(x_0, x_1) \end{aligned}$$

Since $\delta \in [0, 1)$ and $b \succeq I$, we have

$$\lim_{n,m\to\infty} d(x_n, x_m) = \theta.$$

We deduce that the sequence x_n is a Cauchy sequence □

Definition 0.5. [8] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$ metric space and $\{x_n\}$ a sequence in X.

We have:

1) $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to \theta$ as $n \to \infty$.
2) $\{x_n\}$ is a Cauchy sequence if $d(x_m, x_n) \to \theta$ as $m, n \to \infty$
3) $(X, \mathbb{A}, d)$ is complete if very Cauchy sequence in X is convergent.

Definition 0.6. [8] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{A}'_+$ be two mappings.

T is said to be $\alpha-$ admissible if

$$\alpha(x, y) \succeq I \Rightarrow \alpha(Tx, Ty) \succeq I.$$

Definition 0.7. [8] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{A}'_+$ be two mappings such that T is $\alpha-$ admissible.

T is said to be triangular $\alpha-$ admissible if

$$\alpha(x, y) \succeq I \textit{ and } \alpha(y, z) \succeq I \ \Rightarrow \alpha(x, z) \succeq I$$

Definition 0.8. [8] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{A}'_+$ be two mappings.

T is said to be $\alpha-$ orbital admissible if

$$\alpha(x, Tx) \succeq I \ \Rightarrow \alpha(Tx, T^2x) \succeq I$$

Definition 0.9. [8] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{A}'_+$ be two mappings such that T is $\alpha-$ orbital admissible.

T is said to be triangular $\alpha-$ orbital admissible if

$$\alpha(x, y) \succeq I \textit{ and } \alpha(y, Ty) \succeq I \ \Rightarrow \alpha(x, Ty) \succeq I$$

Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$ metric space. We will denote By $\mathcal{CB}(X)$ the set of non-empty bounded closed subsets of X. For $M, N \in \mathcal{CB}(X)$ and $x \in X$, we define

$$d(x, M) = inf_{a \in M} d(x, a) \quad and \quad d(M, N) = sup_{a \in M} d(a, N).$$

The mapping

$$h : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{A}_+$$

given by $h(M,N) = max\{sup_{a\in M} d(a,N), sup_{b\in N} d(b,M)\}$, is the Hausdorff distance between M and N in $\mathcal{CB}(X)$.

A point x is said to be a fixed point of multi-valued mapping $T : X \to \mathcal{CB}(X)$ provided $x \in T(x)$.

In 2014, Hussain et al.[2] introduced a notion of $\alpha-$ completeness for metric spaces.

Definition 0.10. [1] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$ metric space and $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping. The space X is said to be $\alpha-$ complete, if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \succeq I$ for all $n \in \mathbb{N}$ converges in X.

Definition 0.11. [1] Let $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping satisfying the proprety that if

$\alpha(x,y) \succeq I \Rightarrow \alpha_*(Tx, Ty) \succeq I$, where

$\alpha_*(M,N) = inf\{\alpha(x,y) : x \in M, y \in N\}$, then T is said to be α_*- admissible.

Definition 0.12. [1] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$ metric space

and $\alpha, \eta : X \times X \to \mathbb{A}_+$ be two mappings. T is said to be $\alpha - \eta-$ continuous on $(X, \mathbb{A}, d)$, if for given $x \in X$ and a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \succeq I$ $\forall n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$ imply that $Tx_n \to Tx$ as $n \to \infty$.

If $\eta(x_n, x_{n+1}) = I$, then T is said an $\alpha-$ continous mapping.

Definition 0.13. [1] Let $T, S : X \to \mathcal{CB}(X)$ be two multi-valued mappings

and $\alpha : X \times X \to \mathbb{A}'_+$ be a function. Then the pair (T, S) is said to be triangular α_*- admissible if the following conditions hold:

(i) $\alpha(x,y) \succeq I \Rightarrow \alpha_*(Tx, Sy) \succeq I$ and $\alpha_*(Sx, Ty) \succeq I$

(ii) $\alpha(x,y) \succeq I$ and $\alpha(y,z) \succeq I \Rightarrow \alpha(x,z) \succeq I$.

Definition 0.14. [1] Let $T, S : X \to \mathcal{CB}(X)$ be two multi-valued mappings

and $\alpha : X \times X \to \mathbb{A}'_+$ be a function. Then the pair (T, S) is said to be α_*- orbital admissible if the following condition hold:

$\alpha(x, Tx) \succeq I$ and $\alpha_*(x, Sx) \succeq I \Rightarrow \alpha_*(Tx, S^2x) \succeq I$ and $\alpha_*(Sx, T^2x) \succeq I$.

Definition 0.15. [1] Let $T, S : X \to \mathcal{CB}(X)$ be two multi-valued mappings

and $\alpha : X \times X \to \mathbb{A}'_+$ be a function. Then the pair (T, S) is said to be triangular α_*- orbital admissible if the following conditions hold:

(i) (T, S) is α_*- orbital admissible.

(ii) $\alpha(x,y) \succeq I$, $\alpha(y,Ty) \succeq I$ and $\alpha_*(y,Sy) \succeq I \Rightarrow \alpha_*(x,Ty) \succeq I$ and $\alpha_*(x,Sy) \succeq I$.

Lemma 0.16. [1] *Let $T,S : X \to \mathcal{B}(X)$ be two multi-valued mappings such that the pair (T,S) is triangular α_*- orbital admissible.*

Assume that there exists $x_0 \in X$ such that $\alpha_*(x_0,Tx_0) \succeq I$.

Define a sequence $\{x_n\} \in X$ by $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in S(x_{2n+1})$, where $n = 0,1,2,...$

Then $\forall n,m \in \mathbb{N}\cup\{0\}$ with $m > n$, we have $\alpha(x_n,x_m) \succeq I$.

3. Main results

Using C^*- Hausdorff metric on $\mathcal{CB}(X)$ we give a generalization of some common fixed point results for rational contraction of multivalued mappings defined on a C^*-algebra-valued $b-$ metric space.

The following lemmas will be used later.

Lemma 0.17. *Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$ metric space. For any $x,y \in X$ and $M,N,C \in \mathcal{CB}(X)$ we have:*

(i) $d(x,N) \preceq d(x,u)$, *for any* $u \in N$

(ii) $d(x,M) \preceq h(M,N)$

(iii) $h(M,C) \preceq b(h(M,N)+h(N,C))$

(iv) $d(x,M) \preceq b[d(x,y)+d(y,M)]$.

Lemma 0.18. *Let $M,N \in \mathcal{CB}(X)$ such that $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$ metric space and $q \leq 1$. Then, for every $a \in M$ there exists some $u \in N$ such that*

$$qd(a,u) \preceq h(M,N).$$

Proof. If $h(M,N) = \theta$, then $a \in M$ and $qd(a,u) \preceq h(M,N)$ holds for $a = u$.

Suppose that $h(M,N) \succ \theta$.

For any $r \succ \theta$ there exists $u \in M$ such that $d(a,u) \preceq d(a,N)+r \preceq h(M,N)+r$.

We may assume $r = (\frac{1}{q}-1)h(M,N) \succ \theta$, this complete the proof which does not depend on b. □

Now, one can give the definition of $\alpha-$ continuous multivalued mapping.

Definition 0.19. Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space.

Let $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping. Then T said to be a $\alpha-$ continuous multivalued mapping on $(\mathcal{CB}(X), h)$,

if $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \succeq I, \forall n \in \mathbb{N} \cup \{0\}$ and $x \in X$ such that $lim_{n\to+\infty} d(x_n, x) = \theta$ then $lim_{n\to+\infty} h(Tx_n, Tx) = \theta$.

We give the definition of C^*- multivalued contraction.

Definition 0.20. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space with a coefficient $b \succeq I$ a mapping $T : X \to \mathcal{CB}(X)$ is called a C^*- multivalued contraction if there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ and $\|b\|\|\lambda\|^2 < 1$ such that

$$h(Tx, Ty) \preceq \lambda^* d(x, y)\lambda \ \ \forall x, y \in X$$

The following is nontrivial example of C^*- multivalued contraction.

Example 0.21. Let $X = [-1, 1]$, $\mathbb{A} = \mathbb{R}^2$ and $d : X \times X \to \mathbb{A}^+$ given by

$$d(x, y) = (|x - y|, 0) \ \ \forall x, y \in X.$$

It is easy to verify that $(X, \mathbb{A}, d)$ is a C^{*-}algebra valued b metric space with coefficient $(2, 0)$.

Let $M, N \in \mathcal{CB}(X)$ be given by the closed intervals in X as

$$M = [0, \frac{1}{4}] \ \ and \ \ N = [\frac{1}{2}, \frac{3}{4}]$$

Then

$$\begin{aligned} h(M, N) &= max\{sup_{a\in M} d(a, N), sup_{b\in N} d(b, M)\} \\ &= max\{(\frac{1}{2}, 0), (\frac{1}{2}, 0)\} \\ &= (\frac{1}{2}, 0). \end{aligned}$$

Define $T : X \to \mathcal{CB}(X)$ by $Tx = \{y; 0 \leq y \leq \frac{1}{4}x\}$.

Then

$$h(Tx, Ty) \preceq \lambda^* d(x, y)\lambda \ \ with \ \ \|\lambda\| = \frac{1}{2}$$

Hence T is a C^*- multivalued contraction.

We present the following fixed point theorem.

Theorem 0.22. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued $b-$metric space with a coefficient $b \succeq I$ and $T : X \to \mathcal{CB}(X)$ be a C^*- multivalued contraction. That is, there exists $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ and $\|b\|\|\lambda\|^2 < 1$ such that*

$$h(Tx, Ty) \preceq \lambda^* d(x, y)\lambda \;\; \forall x, y \in X$$

Then T has a fixed point.

Proof. Let $x_0 \in X$, consider a point $x_1 \in Tx_0$ and $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \preceq h(Tx_0, Tx_1) + \lambda^*\lambda.$$

Again, since Tx_1 and Tx_2 are closed and bounded subsets of X and x_2 lies in Tx_1 there will be a point $x_3 \in Tx_2$ which satisfies

$$d(x_2, x_3) \preceq h(Tx_1, Tx_2) + (\lambda^*\lambda)^2.$$

Proceeding in this way we obtain a sequence $\{x_n\}_{n\in\{1,2,..\}}$ of points of X such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \preceq h(Tx_{n-1}, Tx_n) + (\lambda^*\lambda)^n \;\; \forall n \geq 1.$$

We note that for all $n \geq 1$

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq h(Tx_{n-1}, Tx_n) + (\lambda^*\lambda)^n \\ &\preceq \lambda^* d(x_{n-1}, x_n)\lambda) + (\lambda^*\lambda)^n \\ &\preceq \lambda^*[h(Tx_{n-2}, Tx_{n-1}) + (\lambda^*\lambda)^{n-1}]\lambda + (\lambda^*\lambda)^n \\ &= \lambda^*[h(Tx_{n-2}, Tx_{n-1})]\lambda + 2(\lambda^*\lambda)^n \\ &\preceq \lambda^{*n} d(x_0, x_1)\lambda^n + n(\lambda^*\lambda)^n \end{aligned}$$

Hence for $\forall n, m \geq 1$

$$\begin{aligned} d(x_n, x_{n+m}) &\preceq b[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+m-1}, x_{n+m})] \\ &\preceq b[\lambda^{*n} d(x_0, x_1)\lambda^n + n(\lambda^*\lambda)^n + (\lambda^{*(n+1)} d(x_0, x_1)\lambda^{n+1} + (n+1)(\lambda^*\lambda)^{n+1} + ... \\ &+ (\lambda^{*(n+m-1)} d(x_0, x_1)\lambda^{n+m-1} + (n+m-1)(\lambda^*\lambda)^{n+m-1}] \\ &\leq b(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + b^2(d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})) \\ &+ ... + b^{m-n+1}(d(x_{n+m-2}, x_{n+m-1}) + d(x_{n+m-1}, x_{n+m})). \\ &= b[\sum_{k=n}^{n+m-1} \lambda^{*k} d(x_0, x_1)\lambda^k + \sum_{k=n}^{n+m-1} (\lambda^*\lambda)^k] \\ &= \sum_{k=i}^{n+m-1} |(b^{\frac{1}{2}} d(x_0, x_1))^{\frac{1}{2}} \lambda^k|^2 + \sum_{k=i}^{n+m-1} |b^{\frac{1}{2}} \lambda^k|^2 \\ &\preceq I\|b\|\|d(x_0, x_1)\| \sum_{k=n}^{n+m-1} \|\lambda^2\|^k + I\|b\| \sum_{k=n}^{n+m-1} \|\lambda^2\|^k \to \theta \; as \; m \to \infty. \end{aligned}$$

It follows that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since is complete, the sequence $\{x_n\}$ will converge to some $x_0 \in X$. Also

$$h(Tx_n, Tx_0) \preceq \lambda^* d(x_n, x_0)\lambda$$

Therefore, the sequence $\{Tx_n\}$ converges to Tx_0. Also $x_n \in Tx_{n-1}$ $\forall n \in \{1, 2, ...\}$ and $d(x_n, Tx_0) \to \theta$ as $n \to \infty$. We obtain that $x_0 \in Tx_0$. □

Definition 0.23. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space.

Let $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping and $T, S : X \to \mathcal{CB}(X)$ two multivalued mappings said to be a pair of generalized rational α_*- contraction type for multivalued mappings if there exists $x, y \in X$ with $\alpha(x, y) \succeq I$ and satisfies

(0.1) $$h(Tx, Sy) \preceq \lambda^* M(x, y)\lambda, for \; \lambda \in \mathbb{A} \; with \; \|\lambda\| < 1 \;\; and \;\; \|b\|\|\lambda\|^2 < 1$$

where

(0.2) $$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2}\}$$

We prove a common fixed point theorem.

Theorem 0.24. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space with $b \succeq I$ and $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping. Let $T, S : X \to \mathcal{CB}(X)$ be a pair of generalized rational α_*- contraction type for multivalued mappings*

(i) *$(X, \mathbb{A}, d)$ is an $\alpha-$ complete*

(ii) *(T, S) is triangular α_*- orbital admissible.*

(iii) *there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \succeq I$ for $x_0 \in X$*

(iv) *T and S are $\alpha-$ continuous.*

Then there exists a common fixed point of T and S in X.

Proof. Let $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \succeq I$. Let $x_1 \in Tx_0$ so that $\alpha(x_0, x_1) \succeq I$ and $x_1 \neq x_0$.

We have

$$0 < d(x_1, Sx_1) \preceq h(Tx_0, Sx_1) \preceq \lambda^* M(x_0, x_1)\lambda$$

there exists $x_2 \in Sx_1$ such that

$$d(x_1, x_2) \preceq h(Tx_0, Sx_1) \preceq \lambda^* M(x_0, x_1)\lambda.$$

With

$$\begin{aligned} M(x_0, x_1) &= max\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Sx_1), \frac{d(x_0, Sx_1) + d(x_1, Tx_0)}{2}\} \\ &= max\{d(x_0, x_1), d(x_0, x_1), d(x_1, Sx_1), \frac{d(x_0, Sx_1) + d(x_1, Tx_0)}{2}\} \\ &= max\{d(x_0, x_1), d(x_0, x_1), d(x_1, Sx_1)\} \\ &= max\{d(x_0, x_1), d(x_1, Sx_1)\}. \end{aligned}$$

If $max\{d(x_0, x_1), d(x_1, Sx_1)\} = d(x_1, Sx_1)$, we get

$$d(x_1, Sx_1) \preceq \lambda^* d(x_1, Sx_1)\lambda$$
$$\Rightarrow \|d(x_1, Sx_1)\| \leq \|\lambda\| \|d(x_1, Sx_1)\|$$

which a contradiction, hence $max\{d(x_0, x_1), d(x_1, Sx_1)\} = d(x_0, x_1)$,

then

$$d(x_1, x_2) \preceq \lambda^* d(x_0, x_1)\lambda.$$

In the same way, for $x_2 \in Sx_1$ and $x_3 \in Tx_2$, we obtain

$$d(x_2, x_3) \preceq h(Sx_1, Tx_2) \preceq \lambda^* M(x_1, x_2)\lambda$$

where

$$\begin{aligned} M(x_1,x_2) &= max\{d(x_1,x_2), d(x_1,Sx_1), d(x_2,Tx_2), \frac{d(x_1,Sx_2)+d(x_2,Tx_1)}{2}\} \\ &= max\{d(x_1,x_2), d(x_2,Tx_2)\}. \end{aligned}$$

If $M(x_1,x_2) = d(x_2,Tx_2)$, by

$$0 < d(x_2,Tx_2) \preceq h(Sx_1,Tx_2) \preceq \lambda^* d(x_2,Tx_2)\lambda$$

we have

$$\|d(x_2,Tx_2)\| < \|\lambda\|\|d(x_2,Tx_2)\|$$

a contradiction, hence

$max\{d(x_1,x_2), d(x_2,Tx_2)\} = d(x_1,x_2)$

and we have $d(x_2,x_3) \preceq \lambda^* d(x_1,x_2)\lambda$.

We define a sequence $\{x_n\}$ by $x_{2n+1} \in Tx_{2n}$ and $x_{2n} \in Sx_{2n+1}$, $n = 0,1,2,....$

So

$$\alpha(x_n,x_{n+1}) \succeq I \ , \ \forall n \in \mathbb{N} \cup \{0\},$$

then

$$0 < d(x_{2n+1},Sx_{2n+1}) \preceq h(Tx_{2n},Sx_{2n+1}) \preceq \lambda^* M(x_{2n},x_{2n+1})\lambda, \tag{0.3}$$

and

$$d(x_{2n+1},x_{2n+2}) \preceq h(Tx_{2n},Sx_{2n+1}) \preceq \lambda^* M(x_{2n},x_{2n+1})\lambda, \tag{0.4}$$

By Lemma 0.17 we have

$$\begin{aligned} \frac{d(x_{2n+1},Tx_{2n})+d(x_{2n},Sx_{2n+1})}{2} &= \frac{d(x_{2n},Sx_{2n+1})}{2} \\ &\preceq b[\frac{d(x_{2n},x_{2n+1})+d(x_{2n+1},Sx_{2n+1})}{2}] \\ &\preceq bmax\{d(x_{2n+1},x_{2n}), d(x_{2n+1},Sx_{2n+1})\}. \end{aligned}$$

Then

$$M(x_{2n},x_{2n+1}) = max\{d(x_{2n},x_{2n+1}), d(x_{2n},Tx_{2n}), d(x_{2n+1},Sx_{2n+1}), \frac{d(x_{2n+1},Tx_{2n})+d(x_{2n},Sx_{2n+1})}{2}\}$$
$$= max\{d(x_{2n+1},x_{2n}), d(x_{2n+1},Sx_{2n+1})\}.$$

If

$$max\{d(x_{2n+1},x_{2n}), d(x_{2n+1},Sx_{2n+1})\} = d(x_{2n+1},Sx_{2n+1}),$$

then from (3.3) we obtain

$$d(x_{2n+1},Sx_{2n+1}) \preceq \lambda^* d(x_{2n+1},Sx_{2n+1})\lambda$$
$$\Rightarrow \|d(x_{2n+1},Sx_{2n+1})\| < \|\lambda\|\|d(x_{2n+1},Sx_{2n+1})\|$$

which is a contradiction,

hence $\{x_n\}$ is a Cauchy sequence. By completeness of $(X,\mathbb{A},d)$, there exists $z \in X$ such

$$\forall n \in \mathbb{N}\cup\{0\}\ lim_{n\to+\infty} d(x_n,z) = \theta$$
$$\Rightarrow lim_{n\to+\infty} d(x_{2n+1},z) = lim_{n\to+\infty} d(x_{2n+2},z) = \theta.$$

Since S is $\alpha-$ continuous, $lim_{n\to+\infty} h(Sx_{2n+2},Sz) = \theta$.

Therefore

$$d(z,Sz) \preceq b[d(z,x_{2n+1}) + d(x_{2n+1},Sz)] \to \theta.$$

So, $z \in Sz$. Similarly, $z \in Tz$.

Thus, z is a common fixed point of T and S.

□

Assuming the following conditions, we prove that Theorem 0.25 still hold for T not necessarily continuous: In the following we show that the α continuity proprety is replaced by a new condition.

Theorem 0.25. *Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b-$metric space with $b \succeq I$ and $\alpha : X\times X \to \mathbb{A}'_+$ be a mapping.*

Let $T, S : X \to \mathcal{CB}(X)$ be a pair of generalized rational α_*- contraction type for multivalued mappings

(i) *$(X, \mathbb{A}, d)$ is an $\alpha-$ complete*
(ii) *(T, S) is triangular α_*- orbital admissible.*
(iii) *there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \succeq I$ for $x_0 \in X$*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \succeq I$ $\forall n \in \mathbb{N} \cup \{0\}$ and $lim_{n\to+\infty} d(x_n, z) = \theta$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \succeq I$ $\forall k \in \mathbb{N} \cup \{0\}$.*

Then there exists a common fixed point of T and S in X.

Proof. Let $\{x_n\}$ be a sequence in X such that $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$, $n = 0, 1, 2, ...$, with $\alpha(x_n, x_{n+1}) \succeq I$ $\forall k \in \mathbb{N} \cup \{0\}$ and $x_n \to z \in X$.

By (iv), we have

$$d(z, Tz) \preceq b[d(z, x_{2n(k)+1}) + d(x_{2n(k)+1}, Tz)] \tag{0.5}$$

$$\preceq bd(z, x_{2n(k)+1}) + bh(Sx_{2n(k)}, Tz) \tag{0.6}$$

$$\preceq bd(z, x_{2n(k)+1}) + b\lambda^* M(x_{2n(k)}, z)\lambda. \tag{0.7}$$

Where

$$M(x_{2n(k)}, z) = max\{d(x_{2n(k)}, z), d(x_{2n(k)}, Sx_{2n(k)}), d(z, Tz), \frac{d(x_{2n(k)}, Sz) + d(z, Tx_{2n(k)})}{2}\}$$

Letting $k \to \infty$, we get $M(x_{2n(k)}, z) \to d(z, Tz)$ and by (3.7) we have

$$d(z, Tz) \preceq bd(z, x_{2n(k)+1}) + b\lambda^* d(z, Tz)\lambda \Rightarrow 1 < \|b\| \|\lambda\|^2$$

which a contradiction.

Then $z \in Tz$ i.e, z is a fixed point of T.

Proceeding in this manner we prove that $z \in Sz$ i.e, z is the common fixed point of T and S. □

We denote Φ the class of all functions $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ such that for any bounded sequence $\{t_n\}$ of positive real numbers, $\phi(t_n) \to I \Rightarrow t_n \to \theta$ and $\|\phi\| < 1$

And Ψ the class of the functions $\psi : \mathbb{A}_+ \to \mathbb{A}_+$ satisfying the conditions:

(i) ψ is nondecreasing and continuous,
(ii) $\psi(t) = \theta \Leftrightarrow t = \theta$

Definition 0.26. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$ metric space with $b \succ I$ and $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping. Let $T, S : X \to \mathcal{CB}(X)$ be a pair of generalized rational $\alpha_* - \psi-$ Geraghty contraction type for multivalued mappings if there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that for $x, y \in X$, with $\alpha(x, y) \succeq I$, the pair (T, S) satisfies the following inequality:

$$\alpha(x,y)\psi(h(Tx,Sy)) \preceq \phi(\psi(M(x,y))).\psi(M(x,y)), \tag{0.8}$$

where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Ty)}{2}\}.$$

Theorem 0.27. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space with $b \succeq I$ and $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping. Let $T, S : X \to \mathcal{CB}(X)$ be a pair of generalized rational $\alpha_* - \psi-$ Geraghty contraction type for multivalued mappings*

(i) *$(X, \mathbb{A}, d)$ is an $\alpha-$ complete*
(ii) *(T, S) is triangular α_*- orbital admissible.*
(iii) *there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \succeq 1$ for $x_0 \in X$*
(iv) *T and S are $\alpha-$ continuous.*

Then there exists a common fixed point of T and S in X.

Proof. Let $x_0 \in X$, construct the sequence $\{x_n\}$ such that $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$, $n = 0, 1, 2, ...$, with $\alpha(x_n, x_{n+1}) \succeq I$ $\forall k \in \mathbb{N} \cup \{0\}$. By (3.8) we have

$$\begin{aligned} 0 < \psi(d(x_1, Sx_1)) &\preceq \psi(h(Tx_0, Sx_1)) \\ &\preceq \alpha(x_0, x_1)\psi(h(Tx_0, Sx_1)) \\ &\preceq \phi(\psi(M(x_0, x_1))).\psi(M(x_0, x_1)) \end{aligned}$$

there exists $x_2 \in Sx_1$ such that

$$\psi(d(x_1, x_2)) \preceq \alpha(x_0, x_1)\psi(h(Tx_0, Sx_1)) \preceq \phi(\psi(M(x_0, x_1))).\psi(M(x_0, x_1)).$$

With

$$\begin{aligned} M(x_0,x_1) &= max\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Sx_1), \frac{d(x_0,Sx_1)+d(x_1,Tx_0)}{2}\} \\ &= max\{d(x_0,x_1), d(x_0,x_1), d(x_1,Sx_1), \frac{d(x_0,Sx_1)+d(x_1,Tx_0)}{2}\} \\ &= max\{d(x_0,x_1), d(x_0,x_1), d(x_1,Sx_1)\} \\ &= max\{d(x_0,x_1), d(x_1,Sx_1)\}. \end{aligned}$$

If $max\{d(x_0,x_1), d(x_1,Sx_1)\} = d(x_1,Sx_1)$, we get

$$\psi(d(x_1,Sx_1)) \preceq \phi(\psi(d(x_1,Sx_1))).\psi(d(x_1,Sx_1)).$$

$$\Rightarrow \|\psi(d(x_1,Sx_1))\| \leq \|\phi(\psi(d(x_1,Sx_1)))\|\|\psi(d(x_1,Sx_1))\|$$

which a contradiction, hence $max\{d(x_0,x_1), d(x_1,Sx_1)\} = d(x_0,x_1)$, then

$$\psi(d(x_1,x_2)) \preceq \phi(\psi(d(x_0,x_1))).\psi(d(x_0,x_1))$$

In the same way, for $x_2 \in Sx_1$ and $x_3 \in Tx_2$, we obtain

$$\psi(d(x_2,x_3)) \preceq \alpha(x_1,x_2)\psi(h(Sx_1,Tx_2)) \preceq \phi(\psi(M(x_1,x_2))).\psi(M(x_1,x_2))$$

where

$$\begin{aligned} M(x_1,x_2) &= max\{d(x_1,x_2), d(x_1,Sx_1), d(x_2,Tx_2), \frac{d(x_1,Sx_2)+d(x_2,Tx_1)}{2}\} \\ &= max\{d(x_1,x_2), d(x_2,Tx_2)\}. \end{aligned}$$

If $M(x_1,x_2) = d(x_2,Tx_2)$, we obtain

$$\psi(d(x_2,x_3)) \preceq \phi(\psi(d(x_2,Tx_2))).\psi(d(x_2,Tx_2)).$$

$$\Rightarrow \|\psi(d(x_2,Tx_2))\| \leq \|\phi(\psi(d(x_2,Tx_2)))\|\|\psi(d(x_2,Tx_2))\|$$

which is a contradiction, hence
$max\{d(x_1,x_2), d(x_2,Tx_2)\} = d(x_1,x_2)$
and we have

$$\psi(d(x_2,x_3)) \preceq \phi(\psi(d(x_1,x_2))).\psi(d(x_1,x_2))$$

.

We define a sequence $\{x_n\}$ by $x_{2n+1} \in Tx_{2n}$ and $x_{2n} \in Sx_{2n+1}$, $n = 0,1,2,....$ So

$$\alpha(x_n,x_{n+1}) \succeq I \ , \ \forall n \in \mathbb{N} \cup \{0\},$$

then

(0.9)

$$\psi(d(x_{2n+1}, Sx_{2n+1})) \preceq \psi(h(Tx_{2n}, Sx_{2n+1})) \preceq \phi(\psi(M(x_{2n}, x_{2n+1})))\psi(M(x_{2n}, x_{2n+1})),$$

and

(0.10)

$$\psi(d(x_{2n+1}, x_{2n+2})) \preceq \psi(h(Tx_{2n}, Sx_{2n+1})) \preceq \phi(\psi(M(x_{2n}, x_{2n+1})))\psi(M(x_{2n}, x_{2n+1})).$$

Where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \\ &\quad \frac{d(x_{2n+1}, Tx_{2n}) + d(x_{2n}, Sx_{2n+1})}{2}\} \\ &= max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Sx_{2n+1})\}. \end{aligned}$$

If

$$max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Sx_{2n+1})\} = d(x_{2n+1}, Sx_{2n+1}),$$

then

$$\psi(d(x_{2n+1}, Sx_{2n+1})) \preceq \phi(\psi(d(x_{2n+1}, Sx_{2n+1}))).\psi(d(x_{2n+1}, Sx_{2n+1})).$$

$$\Rightarrow \|\psi(d(x_{2n+1}, Sx_{2n+1}))\| \leq \|\phi(\psi(d(x_{2n+1}, Sx_{2n+1})))\|\|\psi(d(x_{2n+1}, Sx_{2n+1}))\|$$

which is a contradiction, hence $max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Sx_{2n+1})\} = d(x_{2n+1}, x_{2n})$ and we have

$$\psi(d(x_{2n+1}, Sx_{2n+1})) \preceq \phi(\psi(d(x_{2n+1}, x_{2n}))).\psi(d(x_{2n+1}, x_{2n})).$$

Using propreties of ψ and ϕ we conclud that $\{x_n\}$ is a Cauchy sequence. By completeness of $(X, \mathbb{A}, d)$, there exists $z \in X$ such

$$\forall n \in \mathbb{N} \cup \{0\} \; lim_{n\to+\infty} d(x_n, z) = \theta$$

$$\Rightarrow lim_{n\to+\infty} d(x_{2n+1}, z) = lim_{n\to+\infty} d(x_{2n+2}, z) = \theta.$$

Since S is $\alpha-$ continuous, $lim_{n\to+\infty} h(Sx_{2n+2}, Sz) = \theta$.

Therefore

$$d(z, Sz) \preceq b[d(z, x_{2n+1}) + d(x_{2n+1}, Sz)] \to \theta.$$

So, $z \in Sz$. Similarly, $z \in Tz$.

Then T and S have a common fixed point in X.

□

Assuming the following conditions, we prove that Theorem **??** still hold for T not necessarily continuous: The following theorem is a consequence of the Theorem 0.28 in the case of the generalized rational $\alpha_* - \psi-$ Geraghty contraction type for multivalued mappings.

Theorem 0.28. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued $b-$metric space with $b \succeq I$ and $\alpha : X \times X \to \mathbb{A}'_+$ be a mapping. Let $T, S : X \to \mathcal{CB}(X)$ be a pair of generalized rational $\alpha_* - \psi-$ Geraghty contraction type for multivalued mappings*

(i) *$(X, \mathbb{A}, d)$ is an $\alpha-$ complete*

(ii) *(T, S) is triangular α_*- orbital admissible.*

(iii) *$\alpha_*(x_0, Tx_0) \succeq I$ for $x_0 \in X$*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \succeq I$ $\forall n \in \mathbb{N} \cup \{0\}$ and $lim_{n\to+\infty} d(x_n, z) = \theta$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \succeq I$ $\forall k \in \mathbb{N} \cup \{0\}$.*

Then there exists a common fixed point of T and S in X.

Declarations

Availablity of data and materials

Not applicable.

Competing interest

The authors declare that they have no competing interests.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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References

[1] Ameer E, Arshad M, Shatanawi W, 2017. Common fixed point results for generalized $\alpha_* - \psi-$contraction multivalued mappings in $b-$metric spaces. J. Fixed Point Theory Appl. 19, 3069–3086. https://doi.org/10.1007/s11784-017-0477-2

[2] Hussain N, Kutbi MA, Salimi P, 2014. Fixed Point Theory in $\alpha-$Complete Metric Spaces with Applications, Abstract and Applied Analysis, vol. 2014, Article ID 280817, 11 pages. https://doi.org/10.1155/2014/280817

[3] Kari A, Rossafi M, Massit H, 2022. On the $\alpha - \psi-$Contractive Mappings in C^*-Algebra Valued $b-$Rectangular Metric Spaces and Fixed Point Theorems, Eur. J. Math. Anal. Vol. 2, 11. DOI: https://doi.org/10.28924/ada/ma.2.11

[4] Ma Z, Jiang L, Sun H, 2014. C^*-algebra-valued metric spaces and related fixed point theorems. Fixed Point Theory Appl 2014, 206. https://doi.org/10.1186/1687-1812-2014-206

[5] Massit H, Rossafi M, 2021. Fixed point theorems for $\psi-$contractive mapping in C^*-algebra valued rectangular $b-$metric spaces, J. Math. Comput. Sci., 11, 6507-6521

[6] Murphy GJ, 1990. C^*-Algebra and operator theory, Academic Press, London.

[7] Nadler SB, 1969. Jr.: Multi-valued contraction mappings. Pac. J. Math., 30: 475–488. 10.2140/pjm.1969.30.475

[8] Omran S, Masmali I, 2021. On the $\alpha-\psi-$Contractive Mappings in C^*-Algebra Valued $b-$Metric Spaces and Fixed Point Theorems, Journal of Mathematics, vol. 2021, Article ID 7865976, 6 pages. https://doi.org/10.1155/2021/7865976

[9] Samet B, Vetro C, Vetro P, 2012. Fixed point theorems for $\alpha - \psi-$contractive type mappings, Nonlinear Analysis. Theory, Methods and Applications, vol. 75, no. 4, pp. 2154–2165.