Stability Analysis of Divorce Dynamics Models

Syamsir[1*], Syamsuddin Toaha[2*], Kasbawati[3*]

Abstract

This article examines the mathematical model of divorce. This model consists of four population classes, namely the Married class (M), the population class who experiences separation of separated beds (S), the population class who is divorced by Divorce (D), and the population class who experiences depression or stress due to divorce Hardship (H). This study focuses on the stability analysis of divorce-free and endemic equilibrium points. Local stability was analyzed using linearization and eigenvalues methods. In addition, the basic reproduction number R_0 is provided via the next generation matrix method. The existence and stability of the equilibrium point are determined from R_0. The results showed that the rate of interaction between population M and populations other than H is very influential on efforts to minimize divorce. Divorce can be minimized when the transmission rate is reduced to $R_0 < 1$. Reducing the transmission rate and increasing the rate of transfer from split bed class to married class can turn divorce endemic cases into non-endemic cases. A numerical simulation is given to confirm the analysis results.

Keywords: Divorce Dynamics, Mathematical Model, Equilibrium Point, Stability Analysis, basic reproduction numbers.

1. Introduction

In a marriage, everyone wants a happy, eternal, and prosperous home life, in accordance with the objectives of marriage as stated in Law No.1 of 1974 ([1,9]). However, not everyone can form a family that they aspire to, this is due to divorce, both death divorce, divorce talaq, and divorce based on the judge's decision [1]. The problem of the dynamics of the divorce epidemic is one of the most familiar problems in society. Several previous studies have modeled problems related to the dynamics of this divorce [15]. For example, the mathematical model of divorce dynamics developed by Patience Pokuaa Gambrah, et al [6], which examines the impact of counseling in divorce cases. Then the research conducted by Patience Pokuaa Gambrah and Yvonne Adzadu [7] divided the mathematical model into 3 compartments, namely marriage (M), separated (S), Divorced (D). In this study, the researchers drew a conclusion that the divorce epidemic can not

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only be controlled by reducing the level of contact between marriage and divorce but also increasing the number of marriages that go into separation and educating themselves from divorce which can be useful in fighting the epidemic.

In this research, researchers will make modifications by combining the two models mentioned above and adding some controls to minimize the occurrence of divorce. Researchers will examine the dynamics of divorce with optimal control.

2. Divorce Dynamics Model

The development of the model in this study is an amalgamation of the dynamics of divorce according to Patience Pokuaa Gambrah, et al [6] and Patience Pokuaa Gambrah and Yvonne Adzadu [7] then resolved with optimal control theory ([5, 13,14]). The results of the model development obtained are four compartments, namely Marriage (M), Separated (S), Divorce (D), and Hardship (H). The development of this model adds family role factors or family interference and external factors (Judges) in an effort to suppress divorce.

The assumptions are formed as follows:

1. New recruits who enter the Marriage class (M) with a constant rate of Λ.
2. Interactions between M with other populations except Hardship (H) cause individuals in the M population to move to the Divorce (D) or Separated (S) population.
3. Individuals in population S are susceptible to transferring to population D at a rate of δS.
4. Population D is prone to depression or stress and moves to population H at a rate of D.
5. Individuals in the population S, D, and H are assumed to be able to remarry.
6. Each compartment of the 4 classes in the population has a death rate with a constant rate of μ.

Schematic diagram of the 4 classes in the model with the assumptions above as follows.
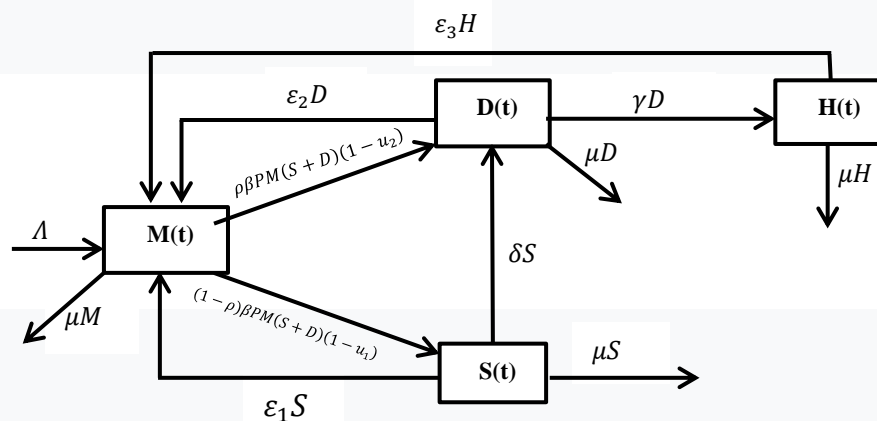

Figure 2.1. Compartment Diagram for Divorce Dynamics Mathematical Model

From the compartment diagram of the model, the following system of differential equations is obtained:

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$$\frac{dM}{dt} = \Lambda + \varepsilon_2 D + \varepsilon_1 S + \varepsilon_3 H - \rho\beta PM(S + D)(1 - u_2) - (1 - \rho)\beta PM(S + D)(1 - u_1) - \mu M$$

$$\frac{dS}{dt} = (1 - \rho)\beta PM(S + D)(1 - u_1) - \varepsilon_1 S - \delta S - \mu S$$

$$\frac{dD}{dt} = \rho\beta PM(S + D)(1 - u_2) + \delta S - \varepsilon_2 D - \gamma D - \mu D$$

$$\frac{dH}{dt} = \gamma D - \varepsilon_3 H - \mu H \tag{2.1}$$

With the total population

$$N = M + S + D + H$$

is assumed that the initial conditions of the system meet:

$$M(0) = M_0, S(0) = S_0, \ D(0) = D_0, \ H(0) = H_0,$$

With $M_0, S_0, D_0, H_0 \geq 0$

From system (4.1) can be obtained:

$$\frac{dN}{dt} = \frac{dM}{dt} + \frac{dS}{dt} + \frac{dD}{dt} + \frac{dH}{dt}$$

$$\frac{dN}{dt} = \Lambda + \varepsilon_2 D + \varepsilon_1 S + \varepsilon_3 H - \rho\beta PM(S + D)(1 - u_2) - (1 - \rho)\beta PM(S + D)(1 - u_1) - \mu M$$
$$+ (1 - \rho)\beta PM(S + D)(1 - u_1) - \varepsilon_1 S - \delta S - \mu S + \rho\beta PM(S + D)(1 - u_2)$$
$$+ \delta S - \varepsilon_2 D - \gamma D - \mu D + \gamma D - \varepsilon_3 H - \mu H$$
$$= \Lambda - \mu(M + S + D + H)$$
$$= \Lambda - \mu N \tag{2.2}$$

The description of the variables and parameters used is described in the following table.

Table 2.1 List of Variables and Parameters of Divorce Dynamics Model

Symbol	Desription	Terms	Unit
$N(t)$	Total Population at one time t.	$N(t) > 0$	Person
$M(t)$	Number of people who are married at one time t.	$M(t) \geq 0$	Person
$S(t)$	Number of people with split-bed status at a time t.	$S(t) \geq 0$	Person
$D(t)$	Number of people who are divorced at one time t.	$D(t) \geq 0$	Person
$H(t)$	Number of people who are divorced and experience depression or stress at any one time t.	$H(t) \geq 0$	Person

Λ	The level of input of individuals who are prone to divorce.	$\Lambda > 0$	Person/year
β	The average interaction between M and individuals other than H.	$\beta > 0$	1/(time x person)
δ	The rate of people divorcing after separation	$\delta > 0$	1/year
γ	The rate at which people experience depression or stress after divorce	$\gamma > 0$	1/year
ε_1	The rate of people getting married again after divorcing	$\varepsilon_1 > 0$	1/year
ε_2	The rate at which people return to married status after separation	$\varepsilon_2 > 0$	1/year
ε_3	The rate at which people return to marriage after experiencing depression or stress	$\varepsilon_3 > 0$	1/year
ρ	The proportion of M who were divorced	$\rho \geq 0$	Person/year
$(1 - \rho)$	The proportion of the number of M separated beds	$(1 - \rho) \geq 0$	Person/year
P	Chances of successful interaction between M (t) and other populations except H (t)	$0 < P < 1$	-
μM	The rate at which the amount of M comes out naturally	$\mu M \geq 0$	1/year
μS	The rate at which the amount of S comes out naturally	$\mu S \geq 0$	1/year
μD	The rate at which the amount of D comes out naturally	$\mu D \geq 0$	1/year
μH	The rate at which the amount of H comes out naturally	$\mu H \geq 0$	1/year
u_1	The proportion of individuals who are spared from being separated from their beds is due to family interference	$0 \leq u_1 \leq 1$	-
u_2	Proportion of individuals who have avoided divorce due to external parties (Judges)	$0 \leq u_2 \leq 1$	-

3. The point of equilibrium and stability is divorce-free and endemic

The divorce dynamics model in model (2.1) can be found for its equilibrium point if it meets $\frac{dM}{dt} = \frac{dS}{dt} = \frac{dD}{dt} = \frac{dH}{dt} = 0$ ([2,3,5,13]). Based on these requirements, system (2.1) is obtained:

$$\Lambda + \varepsilon_2 D + \varepsilon_1 S + \varepsilon_3 H - \rho\beta PM(S + D)(1 - u_2) - (1 - \rho)\beta PM(S + D)(1 - u_1) - \mu M = 0$$

$$(1 - \rho)\beta PM(S + D)(1 - u_1) - \varepsilon_1 S - \delta S - \mu S = 0$$
$$\rho\beta PM(S + D)(1 - u_2) + \delta S - \varepsilon_2 D - \gamma D - \mu D = 0$$
$$\gamma D - \varepsilon_3 H - \mu H = 0 \qquad\qquad (3.1)$$

The divorce-free equilibrium point is a condition where there is no divorce problem in a population or the number of divorced individuals is zero [12]. The point of divorce-free equilibrium is expressed in $E^0 = (M^0, S^0, D^0, H^0)$ by assuming $S^0 = D^0 = 0$. Thus from system (3), the divorce-free equilibrium point is obtained $E^0 = (M^0, S^0, D^0, H^0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$.

The equilibrium point of a divorce or endemic occurs when divorce is still occurring. Thus, endemic occurs when system (1) is stable but all compartments are always positive i.e. $M \neq 0, S \neq 0, D \neq 0$, and $H \neq 0$ or can be expressed in the form $E^* = (M^*, S^*, D^*, H^*)$ where $M^* \neq 0, S^* \neq 0, D^* \neq 0$ and $H^* \neq 0$.

4. Basic Reproduction Numbers

In epidemiology, the rate of spread of an infectious disease is usually measured by a value called the basic reproduction number (R_0) ([4,8,13]).. In this case in order to be free from divorce, it must be set $R_0 < 1$[10, 11]. In this case each divorced individual can only affect on average less than one new individual, so that in the end the divorce will be zero. Meanwhile, if $R_0 > 1$, then each individual who is divorced can cause or affect on average more than one new individual, so that in the end there will be an epidemic.

To find R_0, because what you want to control is the population that causes divorce, only models S and D are needed in equation (2.1), namely:

$$\frac{dD}{dt} = \rho\beta PM(S + D)(1 - u_2) + \delta S - \varepsilon_2 D - \gamma D - \mu D$$
$$\frac{dH}{dt} = \gamma D - \varepsilon_3 H - \mu H$$

From this equation, which causes an increase in divorce cases, namely $\rho\beta PM(S + D) + \delta S$ which can be considered as $\mathcal{F}_i(x)$. Whereas the rest is given as $\mathcal{V}_i(x)$ and negated, so that $\mathcal{F}_i(x)$. and $\mathcal{V}_i(x)$ are obtained as follows:

$$\mathcal{F}_i(x) = \begin{pmatrix} \rho\beta PM(S + D)(1 - u_2) + \delta S \\ 0 \end{pmatrix}$$

$$\mathcal{V}_i(x) = \begin{pmatrix} \varepsilon_2 D + \gamma D + \mu D \\ -\gamma D + \varepsilon_3 H + \mu H \end{pmatrix}$$

Furthermore, we obtain the F and V matrices substituted by the divorce-free equilibrium point $E^0 = (M^0, S^0, D^0, H^0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ with the following results:

$$F = \frac{\partial\mathcal{F}_i(E^0)}{\partial(D, H)} = \begin{pmatrix} \dfrac{\partial F_1}{\partial D} & \dfrac{\partial F_1}{\partial H} \\ \dfrac{\partial F_2}{\partial D} & \dfrac{\partial F_2}{\partial H} \end{pmatrix} = \begin{pmatrix} \rho\beta PM(1 - u_2) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\rho\beta P\Lambda(1 - u_2)}{\mu} & 0 \\ 0 & 0 \end{pmatrix}$$

$$V = \frac{\partial \mathcal{V}_i(E^0)}{\partial(D,H)} = \begin{pmatrix} \dfrac{\partial V_1}{\partial D} & \dfrac{\partial V_1}{\partial H} \\ \dfrac{\partial V_2}{\partial D} & \dfrac{\partial V_2}{\partial H} \end{pmatrix} = \begin{pmatrix} \varepsilon_2 + \gamma + \mu & 0 \\ -\gamma & \varepsilon_3 + \mu \end{pmatrix}$$

Next will be determined the inverse of the matrix V.

$$V^{-1} = \frac{1}{(\varepsilon_2 + \gamma + \mu)(\varepsilon_3 + \mu)} \begin{pmatrix} \varepsilon_3 + \mu & 0 \\ \gamma & \varepsilon_2 + \gamma + \mu \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{(\varepsilon_2 + \gamma + \mu)} & 0 \\ \dfrac{\gamma}{(\varepsilon_2 + \gamma + \mu)(\varepsilon_3 + \mu)} & \dfrac{1}{(\varepsilon_3 + \mu)} \end{pmatrix}$$

So that the basic reproduction value is obtained by multiplying the F matrix by the inverse matrix V as follows:

$$FV^{-1} = \begin{pmatrix} \dfrac{\rho\beta P\Lambda(1 - u_2)}{\mu} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{1}{(\varepsilon_2 + \gamma + \mu)} & 0 \\ \dfrac{\gamma}{(\varepsilon_2 + \gamma + \mu)(\varepsilon_3 + \mu)} & \dfrac{1}{(\varepsilon_3 + \mu)} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\rho\beta P\Lambda(1 - u_2)}{\mu(\varepsilon_2 + \gamma + \mu)} & 0 \\ 0 & 0 \end{pmatrix}$$

Using the characteristic equation $\det(\lambda I - FV^{-1}) = 0$, we get:

$$\det \begin{pmatrix} \lambda - \dfrac{\rho\beta P\Lambda(1 - u_2)}{\mu(\varepsilon_2 + \gamma + \mu)} & 0 \\ 0 & \lambda \end{pmatrix} = 0$$

or

$$\lambda\left(\lambda - \frac{\rho\beta P\Lambda(1 - u_2)}{\mu(\varepsilon_2 + \gamma + \mu)}\right) = 0$$

which has a solution:

$$\lambda_1 = 0$$
$$\lambda_2 = \frac{\rho\beta P\Lambda(1 - u_2)}{\mu(\varepsilon_2 + \gamma + \mu)}$$

Because the value of the basic reproduction number is obtained from the spectral radius or the largest value of the eigenvalues, the basic reproduction number for the system of equations above is

$$R_0 = \frac{\rho\beta P\Lambda(1 - u_2)}{\mu(\varepsilon_2 + \gamma + \mu)} \tag{4.1}$$

with $\mu(\varepsilon_2 + \gamma + \mu) \neq 0$

$R_0 < 1$ happens when $\rho\beta P\Lambda(1 - u_2) < \mu(\varepsilon_2 + \gamma + \mu)$ while $R_0 > 1$ happens when $\rho\beta P\Lambda(1 - u_2) > \mu(\varepsilon_2 + \gamma + \mu)$. Based on the results obtained for R_0, to make $R_0 < 1$, the denominator must be greater than the numerator. Mortality due to natural factors (μ) and the rate of depression or stress after divorce (γ) cannot be increased. Therefore what needs to be done is to increase the number of individuals who return to their married status after separating (ε_2). In addition, the mean interactions of M with other individuals except H (β) and the proportion of M divorced (ρ) must also be derived. Whereas (P) is the chance of successful interaction between M and other populations except H. Thus the divorce rate will be reduced. So it can be said, from this analysis it will be known that the most influential parameters of all the parameters in the divorce dynamic model are the parameters β, ρ, and ε_2. Suppose μ = 0.003, γ = 0.56, ρ = 0.03, P = 1, and Λ = 0.08, then the relationship between the average interaction of M with individuals other than H (β) to the rate of remarriage after parting (ε_2) is illustrated in Figure 2 below.
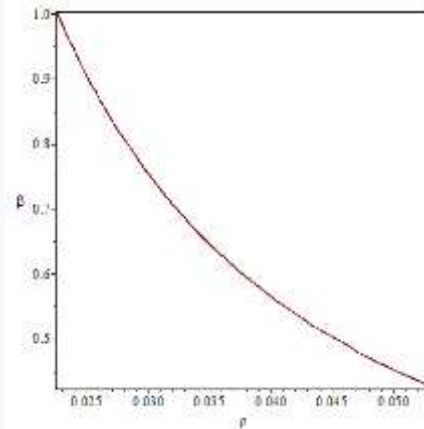

Figure 4.1. The relationship between the functions of β and ε_2 when $R_0 = 1$

5. Numerical Simulation

Simulations are given to provide a geometric picture related to the results of the analysis that has been carried out. Simulations are made using the Maple 18 program and carried out by providing values for each parameter according to the condition R_0.

a. Simulation for $R_0 > 1$

Based on the given model (2.2), and the values adopted from [6] and [7] and with a slight modification for the simulation are given μ = 0.03, β = 0.08, ρ = 0.052, γ = 0.00001, δ = 0.00001, ε_1 = 0.99, ε_2 = 0.28, ε_3 = 0.001, P = 0.021, u_1 = 0.9999, and u_2= 0. With this parameter value, the divorce-free equilibrium point is obtained $E^0 = (M^0, S^0, D^0, H^0) = (4333, 0, 0, 0)$, while the equilibrium point is endemic $E^* = (M^*, S^*, D^*, H^*) = (3546.68, 0.4355, 785.960, 0.2535)$, and $R_0 = 1.221121899$. The eigenvalues associated with the divorce-free equilibrium point are $\{-0.0300, 0.0310, 0.0687, -1.0195\}$. There is a positive eigenvalue at the divorce-free equilibrium point with $R_0 > 1$, it can be concluded that the divorce-free equilibrium point is

unstable. Meanwhile, the eigenvalue is related to the endemic equilibrium point of the divorce $\{-1.0196, -0.0687, -0.0200, -0.0310\}$. All eigenvalues are negative at the endemic equilibrium point of divorce at $R_0 > 1$, it can be concluded that the endemic equilibrium point of divorce is stable. The graphs of the values M^*, S^*, D^* and H^* are presented in Figure 5.1, Figure 5.2, Figure 5.3 and Figure 5.4.
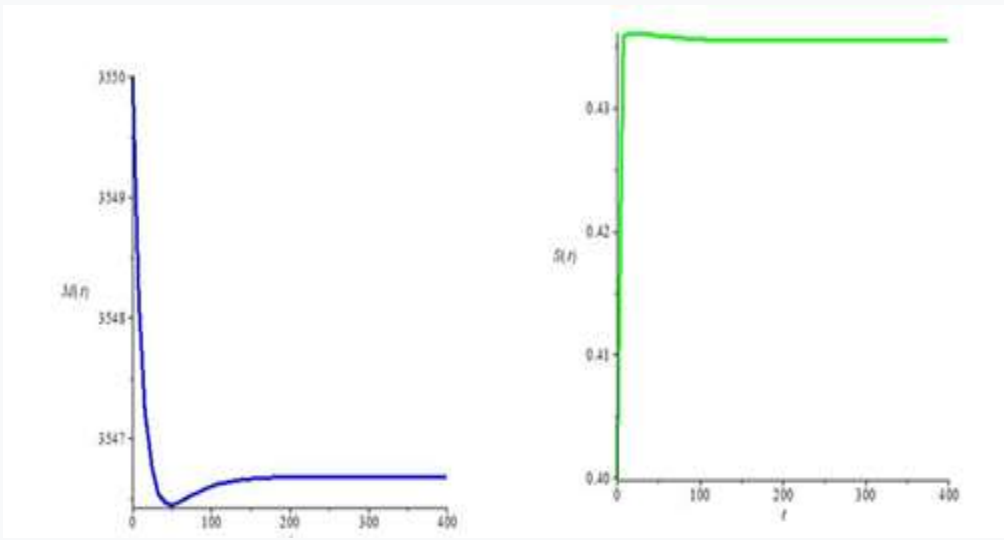

Figure 5.1.Proportion of humans who were married at t for $R_0 > 1$

Figure 5.2. Proportion of humans who were split bed when t for $R_0 > 1$
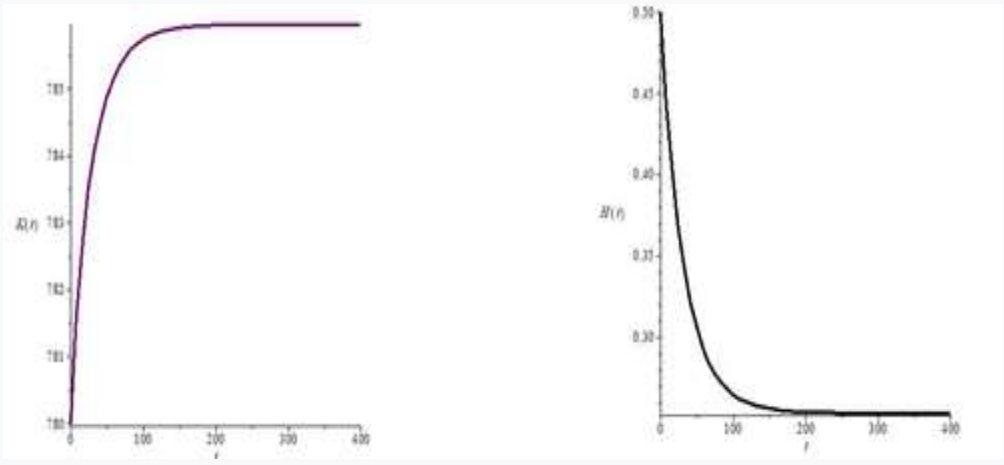

Figure 5.3. Proportion of people who were divorced

Figure 5.4. Proportion of people who were

at t for $R_0 > 1$ were depresse or   stressed at time t for $R_0 > 1$

Figure 5.1 shows that the value $M(t)$ lead to M^* as long as the increments of t. Then in Figure 5.2, Figure 5.3 and Figure 5.4 it can be seen that the value $S(t)$, $D(t)$ and $H(t)$ respectively leading to S^*, D^* and H^* as long as t increments. In other words, Figure 5.1 to Figure 5.4 shows that $\lim_{t \to \infty}(M^*, S^*, D^*, H^*) = E^*$. So the equilibrium point E^* is asymptotically stable. While Graph - a graph of value M^0, S^0, D^0 and H^0 are presented in Figure 5.5, Figure 5.6, Figure 5.7 and Figure 5.8.
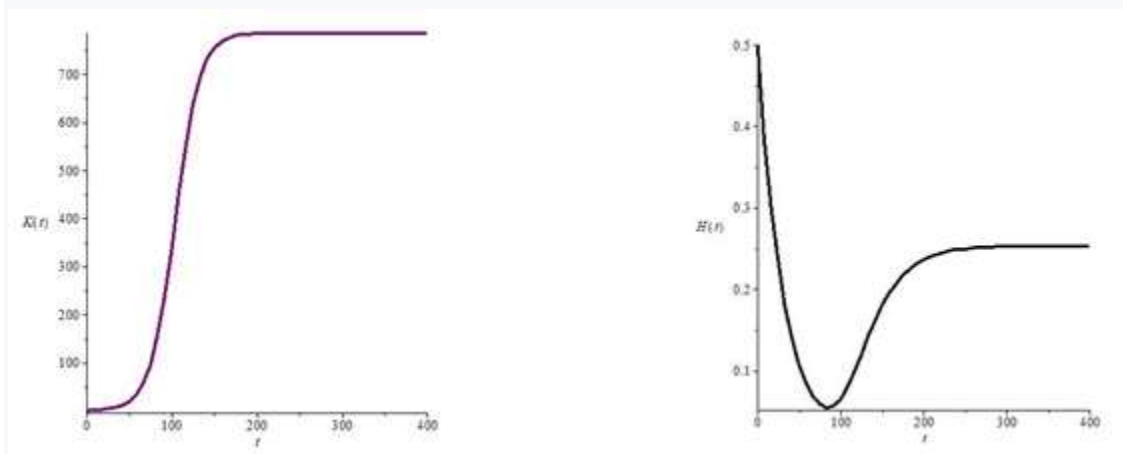

Figure 5.5. Proportion of people who were married at time t for $R_0 > 1$

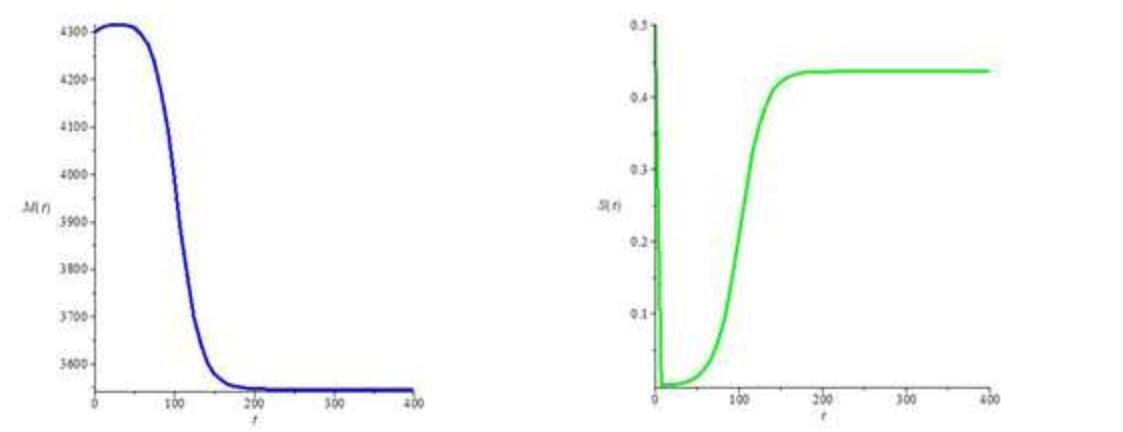**Figure 5.6**. Proportion of people who were separated from bed at time t for $R_0 > 1$

Figure 5.7. Proportion of people who were divorced at time t for $R_0 > 1$

Figure 5.8. Proportion of people who were depressed stressed at time t for $R_0 > 1$

In Figure 5.5, it can be seen that the value of $M(t)$ does not go towards or away from M^0 as long as t increases. Then in Figure 5.6, Figure 5.7 and Figure 5.8 it can be seen that the values of $S(t)$, $D(t)$ and $H(t)$also do not go to S^0, D^0 and H^0 respectively as long as t increases. In other words, in Figures 5.5 to 5.8 it can be seen that $\lim_{t \to \infty}(M^0, S^0, D^0, H^0) \neq E^0$. So the equilibrium point E^0 is unstable.

b. Simulation for $R_0 < 1$

The parameter values given when $R_0 < 1$ are $\mu = 0.03$, $\beta = 0.08$, $\rho = 0.052$, $\gamma = 0.00001$, $\delta = 0.00001$, $\varepsilon_1 = 0.99$, $\varepsilon_2 = 0.35$, $\varepsilon_3 = 0.001$, P = 0.021, $u_1 = 0.9999$, and $u_2 = 0$. With this parameter value, the divorce-free equilibrium point is obtained $E^0 = (M^0, S^0, D^0, H^0) = (4333, 0, 0, 0)$, while the endemic equilibrium point does not exist as well as value $R_0 = 0.9961843109$. The eigenvalues associated with the divorce-free equilibrium point are $\{-0.0300, -0.0310, -0.0012, -1.0195\}$. All e igenvalues are negative at the divorce-free equilibrium point at $R_0 < 1$, it can be concluded that the divorce-free equilibrium point is stable. The graphs of the values M^0, S^0, D^0 and H^0 are presented in Figure 5.9, Figure 5.10, Figure 5.11 and Figure 5.12.
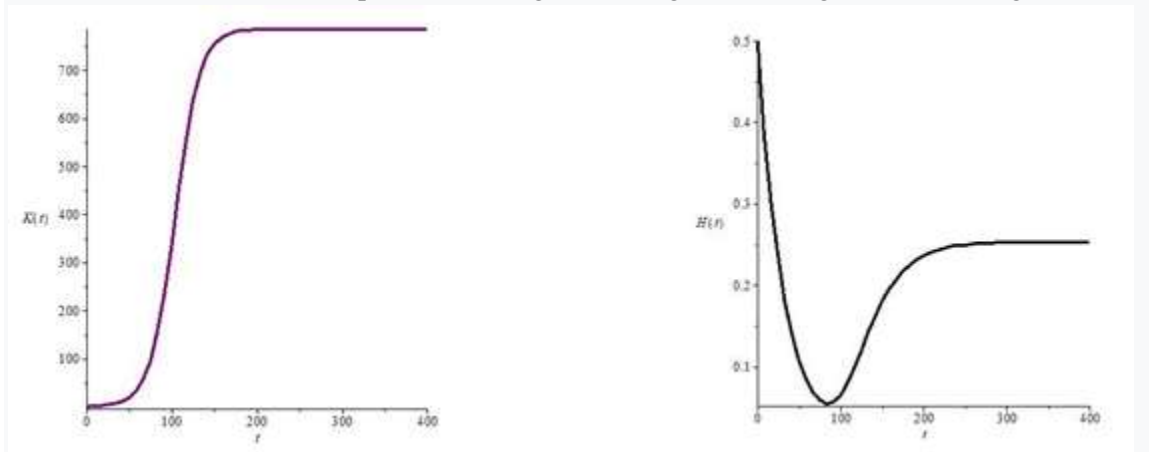

Figure 5.9. Proportion of people who were married at t for $R_0 < 1$ **Figure 5.10**. Proportion of people who were separated from bed at time t for $R_0 < 1$
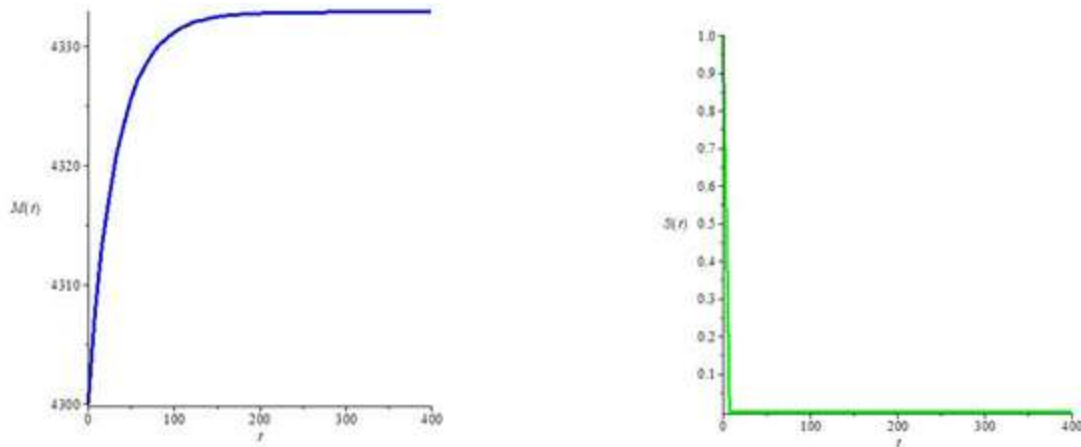
Figure 5.11. Proportion of people who were divorced   **Figure 12**. Proportion of humans who at time t for $R_0 < 1$ were depressed or stressed at time t for $R_0 < 1$

In Figure 5.9 it can be seen that the value of $M(t)$ goes to M^0 as long as t increases. Then in Figure 5.10, Figure 5.11 and Figure 5.12, it can be seen that the values of $S(t)$, $D(t)$ and $H(t)$ go to S^0, D^0 and H^0 respectively as t increases. In other words, in Figures 5.9 to 5.12 it can be seen that $\lim_{t \to \infty}(M^0, S^0, D^0, H^0) = E^0$. So the equilibrium point E^0 is stable for $R_0 < 1$.

6. Conclusion

The mathematical model of the dynamics of divorce has an equilibrium point of divorce-free and divorce. The analysis results from the model also obtained the basic reproduction number (R_0). From the value of the basic reproduction number, it can be seen that whether all the eigenvalues associated with the divorce-free equilibrium point and the existence of divorce are all negative or not. There are several parameters that affect the value of R_0, namely the proportion of married people who are divorced (ρ), the average interaction between people with married status and other individuals (β), the chances of successful interactions between people with married status and other populations (P), the proportion of individuals who have avoided divorce due to the presence of external parties (Judge) (u_2) natural death rate (μ), rate of people returning to marriage status after separation (ε_2) and the rate of people experiencing depression or stress after divorce (γ).
When $R_0 < 1$, all eigenvalues associated with the divorce-free equilibrium point are negative. This means that the point of divorce-free equilibrium is stable. Whereas when $R_0 > 1$, there is one eigenvalue associated with a positive divorce-free equilibrium point and all eigenvalues associated with the equilibrium point of the divorce are all negative.

This means that the divorce-free equilibrium point is unstable and the equilibrium point of divorce is stable. From the results of this analysis and the model simulation performed using maple 18, it is shown that if the reproduction number is smaller than one, there will be no divorce. However, if the parameters ρ, β, P, and u_2 are increased and the parameter ε_2 is decreased, then the value of the reproduction number will be more than one and divorce will occur, in other words there will be a divorce epidemic so that individuals who divorce and interact with other populations will remain in the population.

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